Deciphering Private Equity Incentive Contracting and Fund Leverage Choice

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Abstract

I explain carried interest as a mechanism to induce incentive compatible fund leverage. Fee, leverage and target return data are used to calibrate the model. Steps in the modeling process include developing a tradeoff model of fund capital structure that pits alpha against the costs of financial distress. GPs with convex incentive fee payoff functions limit debt even in the absence of distress costs. LPs optimize by endogenously targeting fund returns. Catch-up fee provisions enable high-skill GPs to extract fees while hitting LP return targets.

Keywords: Mechanism design, Financial intermediation, Private equity, Incentive contracting, Fees, Carried interest, Leverage, Debt, Capital Structure, Real estate

JEL Codes: D86,G21,G23,G32,L14,R33

I. Introduction

What function does the private equity (PE) carried interest contract serve when GPs are endowed with skill and long-term career (reputational) concerns exist to incentivize effort and performance in the shorter-run? To address this question I develop a theory of PE fund capital structure that links directly to incentive fee contracting practices commonly utilized in the PE sector. In essence my argument is that the carried interest contract functions as a complement to indirect compensation incentives, with carried interest used to create appropriate fund capital structuring incentives while also generating acceptable target returns to LP investors.

To motivate model development, I analyze data from the private equity real estate (PERE) sector that exists within the real asset PE category. More specifically, I analyze closed-end Value-Add and Opportunity PERE funds. These funds have gained in prominence over the past 20 years, and now attract more capital than any other category of closed-end funds in the real estate PE part of the market.[1] They are the commercial real estate equivalent of buyout funds, where GP-sponsors take underperforming assets and "turn them around" through repositioning and redevelopment. Focusing on a narrow category of funds that exist within a prescribed asset class helps ensure consistency and comparability in the data, and sharpens assessment when it comes time to calibrate the model.

There are three main categories of interest in my data: incentive fee contract terms, fund leverage levels and fund return targets disclosed in offering documents. I find a median carry hurdle rate of 9.0%, which varies from and is slightly higher than the commonly cited 8.0% rate. There is also small but meaningful variation around the 9.0% median rate, found to be in the 7.0% to 12.0% range. I further document an almost invariant carried interest share of 20.0%, which is consistent with prior findings (e.g., Metrick and Yasuda (2010)). Catch-up fee provisions are not consistently applied in this sector. When they are applied, catch-ups are often set at a 50.0% rate rather than the commonly cited 100% rate. I therefore find that the PERE incentive fee contract is calibrated to a greater extent than previously documented, primarily through the catch-up provision.

[1] See, for example, PREA Survey of Investor Intentions, 2021.

Fund leverage ranges between 50.0% and 75.0% in the data, with an inter-quartile range of 60%-70%, a mean of 63.9%, and a median as well as mode of 65.0%. Net-of-fee target returns (forecasted IRRs) generally range between 13.0% and 20.0%, with a mean of 16.7%. An analysis of the differences between gross- and net-of-fee target returns indicates a fee drag of approximately 3.5% to 4.0%. Finally, I document positive relations between the carry hurdle rate and target returns, as well as between leverage and target returns. This in turn implies a positive relation between the carry hurdle rate and leverage.

For modeling purposes these stylized empirical results lead me to treat the incentive fee contract in a quasi-endogenous manner. To start I take the "9-20" incentive contract as given, and examine the model for fit as it relates to fund leverage and return targets. Then later in the paper I endogenize the GP's incentive contract, primarily in the context of LP return targeting and GP skill heterogeneity. Endogenous LP return targeting results from an optimization that constrains GP net-of-fee fund performance to at least match that obtainable in a setting with zero fees and zero-alpha asset returns. In these cases modeled fund leverage and target returns are verified to reside in empirically documented ranges and in appropriate relation to other variables.

The first step in the modeling process is to develop a model of PE debt costs. In the model, alpha is traded off against costs of financial distress. Debt as a result can be cheap or expensive relative to the frictionless debt cost benchmark. Whenever financial distress costs are positive, there exists an endogenous upper bound on debt funding at which the marginal cost of debt becomes infinite.

The next step in the modeling process is to consider the GP's fund leverage decision. Here the GP's objective is to maximize expected incentive fee payments subject to satisfying participation requirements. I show that fund leverage is set to equalize the marginal costs of debt and preferred equity. Unlike canonical relations in which the cost of equity increases in leverage, in PE the cost of LP (preferred) equity is invariant as a function of leverage. Cheap equity limits fund leverage even when financial distress costs are zero and even though the GP's carried interest payoff function is convex. Incentive compatible fund leverage in this case does not depend on the carried interest share percentage. This separation result helps explain invariance in the carried interest share percentage (almost universally set at 20.0%), as the GP's fund leverage choice problem only depends on the carried interest hurdle rate.

This debt-equity tradeoff raises an issue of how fundraising occurs. In my baseline model I follow Axelson, Stromberg and Weisbach (2009) by raising equity first and then issuing debt at the time of asset acquisition. In this case, an extra dollar of equity implies one dollar less of debt financing. I then consider an alternative fundraising model, whereby the GP raises equity first and then issues debt as a ratchet to fund asset acquisition. Here there are no prescribed limits on fund asset size. This fundraising model creates a different tradeoff, where fund asset size, although larger than in the baseline case, is nevertheless finite. The tradeoff in this case is the marginal cost of debt versus the marginal benefit of increasing the asset pool size, inclusive of the value-enhancing effects of alpha.

The third step in the modeling process is to determine the LP's target net-of-fee returns on committed capital. I then select model parameter values based on a careful analysis of moments and relations documented in the relevant empirical literature. Taking the standard, empirically observed 9-20 incentive fee contract as given, in the base-case the model generates fund leverage of 63.1%. This compares to PERE fund leverage in the data that averages 63.9%. Modeled fund leverage ranges from 59.4% given a low-alpha fund manager to 67.7% when fund assets are low volatility. This range of fund leverage outcomes centers on and closely conforms to empirically observed fund leverage levels. The base-case model also generates net-of-fee target returns that range between 14.6% in the case of a low-skill fund manager to 19.2% in the case of a high-skill fund manager. Fee drag ranges from 3.7% to 4.5%. These return and fee value ranges all match up well with the data.

As a last step to the analysis I augment the baseline carried interest contract to incorporate catch-up fees. This is done in the context of LP return targeting, a process that endogenously determines not only the return target as a constraint on fund returns but also a leverage target that serves as an upper bound on debt funding. When utilizing base-case parameter values, my model generates a return target of 16.72% and a fund leverage target of 64.95%, both of which almost exactly match means (in the case of target return) as well as medians and modes (in the case of fund leverage) documented in the data.

Catch-up fees are shown to further lower the cost of equity capital to reduce fund leverage levels preferred by the GP. Return and fund leverage targets then serve as constraints to be satisfied by the GP who is optimizing carried interest based on the dual choice of fund leverage and the catch-up rate. GP skill heterogeneity is highlighted, with three ranges of outcomes. Low skill GPs are unable to include a catch-up fee provision, since they cannot simultaneously satisfy return target and fund leverage constraints. Moderate skill GPs increase fund leverage to the target upper bound, and calibrate the catch-up rate to just satisfy the return target constraint. High skill GPs charge the full catch-up rate, decreasing fund leverage to further increase fees at the expense of LP returns.

Thus, higher-skill fund managers are able to implement the catch-up fee provision and simultaneously meet the LP's benchmark return requirement, while lower-skill managers cannot. In the PERE data, leverage clusters at 65 percent, with few funds charging the full catch-up rate. This is predicted by my model when alpha is within a limited range above zero, which it is according to Gupta and Van Nieuwerburgh (2021). In contrast, with PE buyout funds in which alpha estimates are higher, full catch-up fee rates are common and fund leverage varies more (see, e.g., Brown et al. (2020)).

This paper relates to several distinct literatures that have emerged in PE and alternative investments. One strand focuses on learning about GP skill and *indirect* incentive compensation that happens when "good current performance increases future inflows of capital, leading to higher fees" in the future (see, e.g., Chung, et al. (2012) addressing PE and Lim, Sensoy and Weisbach (2016) addressing hedge funds). In these studies indirect compensation is found to be at least as important as current direct compensation. But the relative importance of indirect compensation actually deepens the carried interest compensation puzzle, since it implies even less need to provide *any form* of incentive compensation on current fund performance. My model, which links incentive compensation to fund capital structure choice, provides a bridge to the longer-run problem of maximizing lifetime earnings through performance. This in turn allows us to better understand how the GP optimizes fee income over its *entire* lifetime. It does so through two different channels: 1) The fund performance channel that creates future fee income, and 2) The fund capital structure channel that satisfies contemporaneous LP return requirements and optimizes incentive fee payments with respect to the current fund.

A second strand relates to models of alternative investment that prominently feature both fee and fund capital structure in the analysis. In this regard I most directly draw from Metrick and Yasuda (MY, 2010), Axelson, Stromberg and Weisbach (ASW,2009), Lan, Wang and Yang (LWY,2013) and Sorensen, Wang and Yang

(SWY,2014). As with MY, ASW and SWY, I consider a closed-end (finite life) PE fund structure, whereas LWY focuses on hedge funds with an open-end (infinite life) structure. I follow LWY by incorporating both alpha and costs of fund liquidation into analyzing optimal debt funding policy. LWY is, however, a dynamic model that incorporates high-water marks whereas my setting is more tractable, focused on the closed-end PE carried interest contract. In some ways my model structure is closest to SWY, where I extend their baseline closed-end fund model by incorporating costs of financial distress as well as endogenizing fund capital structure. SWY's focus is more on the effects of illiquidity risk on fund valuation, whereas my emphasis is more on explaining fund leverage and how it links to various components of the incentive fee compensation contract. ASW go further than I do by fully endogenizing the convex compensation structure in PE, as well as showing how fundraising/fund capital structure can be used as a commitment device. In comparison, I closely follow their prescribed fundraising structure and instead focus on developing a model that can be calibrated to the data to explain observed fund capital structures as they depend on the compensation contract.

A third strand of literature starts with Carpenter (2000) and Ross (2004), who show there are limits to risk ratcheting with convex compensation contracts when managers are risk averse.[2] Panageus and Westerfied (2009) and Lin, Wang and Yang (2013) extend the result to consideration of high-water mark compensation contracts with risk neutral agents and infinite time horizons. In my extended model of PE fundraising that allows for leverage ratcheting, I show there are limits to fund size, and hence debt in the fund's capital structure, even when managers are risk neutral, not impatient, exposed to no costs of financial distress, optimizing over a finite horizon, and are compensated based on a standard convex carried interest contract. The result shows the parameters of the convex compensation contract are, in fact, incidental, as long as there is some proportional sharing of residual cash flows. What matters is the marginal cost of debt, which increases without bound with fund leverage, versus returns on new investment funded by the debt, which are always finite.

Another strand of the literature focuses on return targeting by LPs, which I fold into both my empirical and theoretical analysis. Return targeting reflects a primary focus by LPs on absolute investment return with less

[2] There is an earlier literature that allows for hedging the compensation contract, showing that the fund managers have incentives to implement as much leverage as possible. See, e.g., Grinblatt and Titman (1989).

focus on the risks of investment.[3] Those who have weighed in on the breakdown of canonical risk-return relations in PE include Gompers, Kaplan and Mukharlyamov (2016), Korteweg (2019), Andonov, Bauer and Cremers (2017), Andonov and Rauh (2019) and Bodnaruk and Simonov (2016). I add to the literature by developing a method in the spirit of Berk and Green (2004) that endogenously generates both a return and fund leverage target for LPs that compete for investment opportunities. These targets function as constraints that must be satisfied by the GP in the design of its compensation contract.

The closest paper to mine in the PERE literature is Shilling and Wurtzbach (2012), who conduct an empirical analysis of PE real estate funds. They find that managers target returns and that fund leverage is "high regardless of market conditions." My model similarly incorporates return targeting by LP's, and generates highly consistent results. But my model offers significantly more structure that directly links compensation to fund leverage to explain the "consistently high leverage" observed with Value-Add and Opportunity PERE funds.

The organization of the paper is as follows. In section II I present and analyze new data on PERE Value-Add and Opportunity funds. In section III I introduce the formal model, first developing a theory of debt pricing that pits alpha against costs of financial distress. In section IV I introduce incentive fees and the GP's leverage choice problem. LP return measurement is considered in section V, followed by a careful selection of model parameters to calibrate and assess the model. LP return targeting, GP skill heterogeneity and the catch-up fee provision are analyzed in section VI. The paper concludes in section VII.

II. Preliminaries

II.A. Some New Empirical Facts and Relations

PE incentive contracting fee structure and its apparent lack of meaningful variation across funds and GP-sponsors remains a puzzle.[4] One reason for this is that detailed incentive fee and fund leverage data have been

[3] Axelson, Sorensen and Stromberg (2014) express return targeting in a slightly different way, referring to it as the "β puzzle". They state: "These studies suggest that buyout funds can acquire regular companies with equity β around 1.0 and then increase their leverage six-fold, yet leave systematic risk unchanged."
[4] See, e.g., Metrick and Yasuda (2010) and Lim, Sensoy and Weisbach (2016).

hard to come by, particularly in combination. My plan in this section is introduce new data on fee structures, capital structures and performance targets for closed-end private equity real estate (PERE) funds.

Why analyzed PERE fund data? Assets that populate PERE funds are commercial real estate properties that are relatively homogeneous in terms of their risk-return characteristics. Fund managers also rely primarily on secured non-recourse debt that is collateralized by fund assets as a source of debt funding, which simplifies the analysis. There are two categories of PERE funds that resemble buyout funds: Value-Add and Opportunity funds. Assets comprising these funds are often described as requiring asset repositioning and refurbishing, as well as significant planning, development and leasing, along with elevated operating risks. Although Value-Add funds are considered somewhat less risky than Opportunity funds from an investment and operating perspective, they have performed similarly over time, deploy similar levels of fund leverage and attract similar types of institutional capital for fund investment.

In focusing on closed-end PERE funds, I draw from two separate data sources. My first data source is *Real Estate Alert*.[5] These data provide information on the components of the GP's fee contract, as well as the fund's return target as measured by forecasted IRR. Fund vintage dates range from 2004 through 2020.[6] Summary statistics on management fees, the carried interest hurdle rate, the carried interest share percentage, and the catch-up rate are reported in columns (1) through (4) of Table 1.

Table 1 - Summary Statistics: *Real Estate Alert Data*

Statistic	Mgmt Fee (1)	Carry Hurdle (2)	Carry Interest (3)	Catch-Up Rate (4)	Target IRR (5)
N	254	254	254	254	254
Mean	1.56%	8.92%	20.4%	41.0%	16.71%
Median	1.50%	9.00%	20.0%	50.0%	16.00%
S.D.	0.27%	1.31%	2.64%	30.4%	2.69%
Min	0.40%	6.00%	10.0%	0.00%	12.00%
Max	2.50%	20.00%	40.0%	100.00%	25.00%

[5] *Real Estate Alert* is wholly owned by Green Street, an independent REIT analysis firm located in southern CA.
[6] Open-end funds and funds with non-US investors were eliminated from the sample. This left 282 observations with management fee and target IRR information, and 254 observations with carry hurdle, carried interest share and catch-up rate information. I trimmed the sample to 254 observations with consistent information across all five variables.

Management fees cluster fairly tightly around 1.50%. Interestingly, the carry hurdle in these data centers on 9.0%, rather than 8.0%, as commonly cited in the literature. My discussions with industry participants indicate that some believe the 9.0% hurdle value derives from long-run before-fee average returns to the NCREIF index of open-end core CRE funds. Carry hurdle values also display some variation. The minimum and maximum carry hurdle values are 6.0% and 20.0%, respectively. One hundred and two out of 254 observations (40.2%) are at a 9.0% carry hurdle, 67 observations (26.4%) at 8.0% and 55 (21.7%) at 10.0%.

In comparison, consistent with findings of Metrick and Yasuda (2010) and others, the carried interest share centers at 20.0% with little variation around the median value. In particular, 235 of 254 funds (92.5%) specify 20.0% as the carried interest share. Finally, the literature suggests that most PE funds employ 100% catch-up provisions (Metrick and Yasuda (2010), Robinson and Sensoy (2013)). In contrast, in this PERE fund data I observe fewer 100% catch-up provisions, as well as substantial variation in catch-up rates. Sixty-seven out of 254 funds (26.4%) do not use a catch-up provision at all, while 94 (37.0%) are set at 50% and only 14 (5.5%) are set at 100%.

Column (5) of Table 1 contains summary statistics for the net-of-fee target IRR. Target returns are set prior to the start of fund operations. Here the mean target return is 16.71%, with a minimum of 12.00% and a maximum of 25.00%. The vast majority of observations (86.6%) are between 14.0% and 20.0% (inclusive).

My second source of data is *Preqin*. *Preqin* is a well-known PE data provider. I was able to obtain select information from *Preqin* on PERE Value-Add and Opportunity fund leverage, a variable missing from the *Real Estate Alert* data. I was also able to obtain detailed information on IRR targets on both a gross-of-fee and net-of-fee basis, which allows me to estimate fee drag as the difference between targeted gross-of-fee and net-of-fee returns. Fund vintage dates are available from 1994 through 2020. However, to enhance comparability with the *Real Estate Alert* data I only consider funds with vintage dates from 2004-2020. I also trim two funds that are significant outliers with respect to fund leverage, eliminating one fund with 0.0% leverage and one fund with 90.0% leverage. This leaves me with a total of 1061 observations. These data, however, contain a large number of missing observations scattered across all variables, which results in much smaller samples of matched (and even unmatched) observations.

Table 2 reports summary statistics for the *Preqin* data. Columns (1) and (3) contain leverage (debt-to-total fund asset book value) results, while columns (2) and (4) contain fee drag results. The difference between results reported in columns (1) and (2) versus (3) and (4) is that the former is based on unmatched sample data, whereas the latter rely on matched sample data.

Table 2 – Summary Statistics: *Preqin* Data

Statistic	Leverage Unmatched (1)	Fee Drag Unmatched (2)	Leverage Matched (3)	Fee Drag Matched (4)
N	364	361	115	115
Mean	63.96%	3.76%	62.37%	3.80%
Median	65.00%	3.50%	65.00%	4.00%
S.D.	8.47%	1.69%	9.03%	1.66%
Min	20.00%	1.00%	25.00%	1.00%
Max	80.00%	16.00%	80.00%	10.20%

Notes: Fee drag is the difference between what *Preqin* labels gross-of-fee and net-of-fee maximum IRR estimated at the fund start date.

Fund leverage ranges from 20.0% to 80.0%, with a median as well as modal value of 65.0% (see Table 3). The mean of the distribution is 63.96% using unmatched data (62.37% using only matched data). Leverage Two hundred and sixty-one of 364 funds (71.7%) report leverage in the range of 60.0% to 70.0%, while 331 funds (90.9%) report leverage in the range of 50.0% to 75.0%.

Table 3 – Sample Distribution of PERE Fund Leverage

Fund Leverage	*20-39%*	*40-49%*	*50%*	*51-59%*	*60%*	*61-64%*	*65%*	*66-69%*	*70%*	*71-75%*	*80%*	*Total*
N	3	6	34	15	54	2	131	13	61	36	9	364

Columns (2) and (4) of Table 2 report fee drag estimates, which equals the difference between gross-of-fee and net-of-fee IRR estimates. The *Preqin* data provide maximum as well as minimum target-IRR estimates. I use maximum estimates to streamline the analysis. Based on mean and median values from both the unmatched and matched samples, 3.50% to 4.00% fee drag is observed.

Lastly, I consider relations between key variables. Using the *Real Estate Alert* data, I calculate a correlation between the carry hurdle value and target IRR of .37, which is significantly different from zero at the 1.0% level. The positive relation is intuitive. All else equal, a higher carry hurdle value implies lower GP fees and therefore higher net-of-fee IRRs for the LP. Using the *Preqin* data, I calculate the correlation between fund leverage and the maximum net-of-fee IRR target. In this case, based on a sample size of 323 a correlation coefficient of .17 results, which is also significantly different from zero at the 1.0% level. A positive relation between leverage and target-IRR is also intuitive. All else equal, LP's in more levered funds should expect higher returns.[7] By inference, then, a positive relation between the carry hurdle value and target return, as well as between leverage and the return target, implies a positive relation between the carry hurdle value and leverage. This relation follows from the GP's endogenous fund leverage choice when confronted with the incentive fee contract, where a higher break-point incentivizes the GP to increase leverage in order to reduce its reliance on more costly equity capital.

In summary, the following empirical facts emerge from this analysis of PERE-based buyout funds. In terms of fee structure, best point estimates are management fees of 1.5%, a carry hurdle rate of 9.0%, and a carried interest share of 20.0%. The carry hurdle finding of 9.0% is higher than the commonly cited 8.0% rate. There is also some variation around the 9.0% median value. Importantly, catch-up fees vary significantly across the sample. I therefore find that the PE incentive fee contract is calibrated to a greater extent than previously documented (as in Metrick and Yasuda (2010), for example), particularly through the catch-up provision.

Fund leverage primarily ranges between 50.0% and 75.0%, with a central tendency in the 60.0% to 70.0% range. Fund leverage tends to cluster at 65.0%. Net-of-fee target-IRRs generally fall between 14.0% and 20.0%, with a mean value of 16.71%. An analysis of the differences between gross- and net-of-fee target-IRRs indicates a fee drag of 3.5% to 4.0%. Fee drag of these magnitudes is less than the 5.0% estimate of Phalippou and Gottschalg (2008) and Robinson and Sensoy (2013) for PE funds, but is close to the estimates of Metrick and Yasuda (2010) for PE as well as Ben-David et al. (2020) in the case of hedge funds. Finally, positive relations between the carry hurdle value and target IRR, as well as leverage and target IRR are documented.

These stylized empirical facts serve as a foundation for model development and evaluation, and will be closely referenced as I move forward in the paper.

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[7] See Brown et al. (2020) for empirical evidence of a positive relation between PE fund leverage and performance.

II.B. Model Sketch

GPs exert effort to establish professional reputation and maximize lifetime earnings. This long-term, lifetime earnings perspective is known in the literature as indirect compensation (e.g., Chung, et al. (2012) and Lim, Sensoy and Weisbach (2016)). Indirect compensation creates strong incentives to exert effort on live funds regardless of compensation structure. The implications of indirect GP compensation incentives are two-fold: 1) Alpha can be taken as a (constant) measure of GP skill that is known to market participants based on past performance, and 2) It deepens the carried interest incentive fee puzzle, as fixed compensation that varies directly with GP skill would seem sufficient.

This leads me to focus on current fund structure for a complementary explanation of carried interest. I specifically focus on analyzing the link between the carried interest fee contract and fund leverage. Given commonly observed interest carry contract terms, I am especially interested in matching model-generated fund leverage level, as well as matching up net-of-fee target returns and fees, with the data. Later I consider the issue of explaining the incentive contract itself when the LP endogenously targets returns and fund leverage through an optimization.

A high level summary of the model structure is as follows. There are three distinct agents considered: 1) The *LP*, which possesses residual cash flow rights, including priority on preferred interest, based on its equity investment in the fund; 2) The *GP*, which possesses day-to-day investing, financing and operating control rights over the fund, as well as contingent residual cash flow rights through the incentive fee contract; and 3) The *secured lender*, which possesses absolute priority cash flow rights by providing debt financing on a limited liability basis. The incentive fee contract that governs profit-sharing is jointly determined by the LP and GP, while the debt contract is negotiated by the GP on behalf of fund investors.

Conditional on the incentive fee contract, the GP maximizes expected incentive fee payoffs by choosing how much debt to use to finance investment. Based on this choice, a secured non-recourse debt contract is executed between the GP and the lender. Debt funding is released at the time of investment, with the acquired assets providing security for the loan.[8] Expected returns from LP investment are measured by the net-of-fee IRR. In the full model examined in section VI, the LP sets a return and fund leverage target endogenously, which constrains the GP's leverage-based optimization problem.

Equity commitment, real asset investment and secured debt financing conditional on a commitment to invest occur in the order described. For modeling purposes these action steps are compressed to the start date of the fund, time t=0, with actions determined simultaneously, in reverse order, based on backward induction. LP equity contribution at t=0 is referred to as committed capital. This capital includes management fees to be paid to the GP over the life of the fund.[9] For modeling purposes I treat all real assets that populate a fund as one large asset. A closed-end fund structure is assumed, with no interim cash flows generated by the fund. Assets are held for T years and then sold, with proceeds immediately distributed at that time based on priority according to the debt and incentive fee compensation contracts.

III. Step 1: The Cost of Debt Financing

III.A. The Case for Incorporating Costs of Financial Distress

Models of PE valuation and capital structure have mostly sidestepped financial distress costs to focus on other frictions.[10] The lack of attention on financial distress costs in PE is surprising (at least to me). It may in part follow from some of the findings of Andrade and Kaplan (1998), who show that high debt levels, and not operating inefficiencies, are the sources of distress in their LBO sample. They also find that financial distress is resolved with fewer losses on average relative to a non-treated sample, and that financial distress costs are

[8] See Axelson, Stromberg and Weisbach (2009) for more on this fundraising timing issue, where they note, "Typically these [PE] funds raise equity capital at the time they are formed, and raise additional capital when the investments are made… [where] this additional capital usually takes the form of debt when the investment is collateralizable."
[9] According to Metrick and Yasuda (2010), committed capital is generally defined as invested capital (the equity contribution) plus lifetime management fees plus establishment cost. I will ignore establishment cost. Lifetime management fees in my model are delivered by the LP into a trust account at t=0, to be used to fund the distribution of management fees throughout the life of the fund. See Arnold, Ling and Naranjo (2017) for an empirical examination of committed funds to PERE that are waiting to be called.
[10] A noteworthy exception is Lan, Wang and Yang (2013) in their examination of hedge funds. They develop an infinite horizon fund valuation model with liquidation costs that impact the cost of debt financing and dynamic fund leveraging decisions.

nearly non-existent for LBO transactions that do not experience a negative shock. Yet, in the end, Andrade and Kaplan estimate costs of financial distress to be in the 10 to 20 percent range.

These results were generated from a sample of buyout funds in the 1980s and 1990s. During that period buyout funds performed very well on average, with high alphas helping offset the usual costs of financial distress. But what happens when alpha declines, as it has in recent years in buyout funds (see, e.g., Phalippou (2021), Gupta and Van Nieuwerburgh (2021)), or when other fund types such as real assets are considered where alphas are not likely to be large to begin with?

In the commercial real estate sector, publicly listed firms and non-institutional PE market alternatives exist to compete directly with institutional PERE. This implies diminished marginal operating, governance and financial engineering gains attributable to PERE, resulting in relatively lower alphas (Gupta and Van Nieuwerburgh (2017), Pagliari (2020), Riddiough (2021)). Furthermore, in their analysis of insurance company loans backed by income-producing collateral, Brown et al. (2006) document distress costs in commercial real estate lending on the order of 20-30% above and beyond losses attributable to the asset's internal transfer value at the time of foreclosure. These costs include addressing deferred maintenance and realizing fire-sale discounts when disposing of the asset. For PERE Value-Add and Opportunity funds, which focus on real estate development and repositioning opportunities, I would expect lender losses to be meaningfully higher than those found in Brown et al. (2006).[11]

III.B. Model

In determining the cost of closed-end PE fund debt, I adopt and extend the baseline model structure of Sorensen, Wang and Yang (2014). The model accounts for alpha as it impacts PE fund performance. Alpha is GP and

[11] Tax shield effects in PE appear to be less important than standard corporate financial analysis might suggest. For example, Jenkinson and Stucke (2011) find that incremental tax shield benefits to the issuance of buyout debt largely accrue to preexisting shareholders through the acquisition share price. In addition, not all PE debt is issued at the target firm (Op-Co) level, with increasing debt in recent years being issued at the fund or sponsor level, presumably with no tax shield pass-through benefits. Similarly, in PERE there is no taxation at the property-firm level to result in double taxation on equity. Even listed commercial real estate firms, REITs, are not taxed at the firm level. Furthermore, the vast majority of LP equity investors in PERE (as well as certain other forms of PE) are tax-exempt institutions such as pension funds, endowments and sovereign wealth funds.

possibly fund specific, persistent over time, and unrelated to general economic conditions. My innovation is to incorporate deadweight costs of financial distress into the model.

Denote the total acquisition cost (book value) of assets that populate the fund as V_0.[12] The equilibrium rate of return to the assets is μ. Asset value V_t evolves continuously over time according to geometric Brownian motion, with a drift of $\mu+\alpha$. Positive alpha causes super-normal returns that accrue to the fund's asset value, resulting from GP operating skill, with the effects emitted continuously and constantly over the life of the fund.[13] Asset volatility is denoted by σ.

The collateralized loan is structured as zero-coupon debt. There is recourse only to the fund's assets in case of default. Default will only occur at the maturity date, T, happening only if $V_T < B$, where B is the face amount of the debt due at maturity. I will not consider any strategic bargaining that could occur between the lender and the fund investor over the deadweight costs incurred by the lender as a result of investor default. Deadweight costs incurred by the lender in the case of PE fund default are proportional to V_T, with the cost parameter denoted by k, $0 \leq k \leq 1$. This implies that, conditional on default, the lender recovers $(1-k)V_T$ at time T.

Lending markets are perfectly competitive. Moreover, as in Sorensen et al. (2014), loans are held by diversified investors with an ability to insure against all relevant state-contingent outcomes. Debt value at the time of issuance equals D_0, determined as follows:

$$D_0 = e^{-rT}\mathbb{E}_0[Min\{\tilde{V}_T\text{^}(1-k)\tilde{V}_T, B\}] \tag{1a}$$

[12] It is not necessary that V_0 is exogenously specified. I will circle back to this issue in the next section.

[13] Sorensen et al. (2014) also make a distinction between full versus incomplete spanning. I follow their baseline model, assuming full spanning. In this regard they state, "Under full spanning the risk of the PE assets is traded in the market, but the PE asset can still earn a positive alpha. In contrast, under complete markets this alpha would be arbitraged away. In our model this arbitrage does not happen because the GP generates the alpha, and the LP can only earn it by investing in the PE fund along with the associated costs. While the LP can dynamically hedge the risks associated with the PE asset, the LP cannot invest in the PE asset directly, and the market is formally incomplete. Depending on the relative bargaining power, a skilled GP may capture some or all of the excess return through the compensation contract, as long as the LP remains willing to invest." I further note that full spanning in PERE has been directly supported in recent empirical work of Goetzmann, Gourier and Phalippou (2019) and Gupta and Van Nieuwerburgh (2021).

where r denotes the riskless rate of interest and $\tilde{V}_T \wedge (1-k)\tilde{V}_T$ indicates that default occurs when $V_T < B$, with recovery equaling $(1-k)V_T$. For valuation purposes the asset drift rate in this case is subject to the usual equivalent martingale adjustment.

Solving this equation is straightforward, with debt value written as follows:

$$D_0 = e^{-rT} BN[d_2] + (1-k)V_0 e^{\alpha T} N[-d_1] \tag{1b}$$

where $N[\cdot]$ denotes the cumulative standard normal distribution function, with

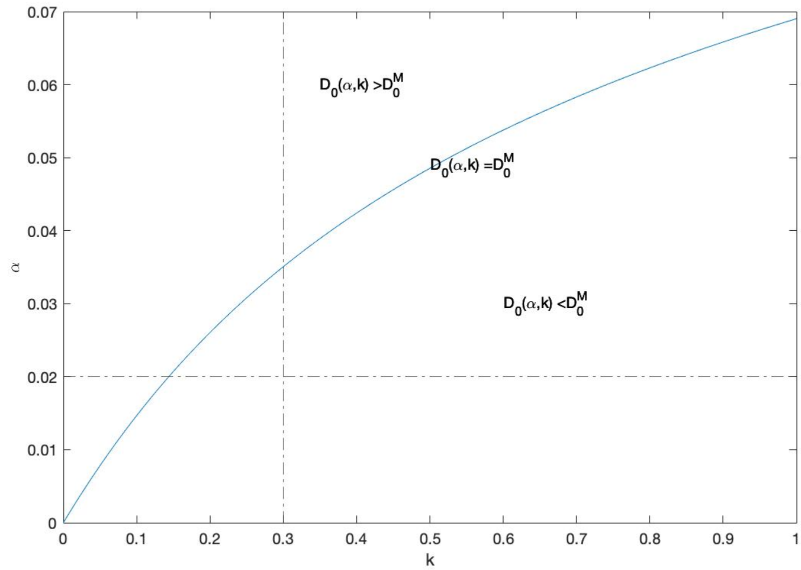
$$d_1 = \frac{ln\left[{V_0}/{B}\right] + \left((r+\alpha) + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},\ d_2 = d_1 - \sigma\sqrt{T} \tag{1b'}$$

Observe that debt value, D_0, reverts to the frictionless debt value analyzed in Merton (1974) when $k = \alpha = 0$. Further note that D_0 decreases linearly in k, the proportional costs of financial distress, whereas D_0 increases in α given that $\frac{\partial D_0}{\partial \alpha} = V_0 e^{\alpha T}\sqrt{T}\left[(1-k)\sqrt{T}N[-d_1] + kn(d_1)/\sigma\right] > 0$, with $n(\cdot)$ denoting the standard normal *pdf.* Finally, note that D_0 retains the important property that it is linearly homogeneous in V_0 and B.

Debt value, D_0, will be smaller or larger (i.e., more expensive or cheap) than the Merton (1974) frictionless debt value, D_0^M, depending on the size of α relative to k. In Figure 1, D_0 is compared to D_0^M given variation in k and α. The solid line indicates (k,α) combinations that result in $D_0 = D_0^M$. For a given cost of financial distress parameter, k, an alpha larger than that indicated along the solid line is necessary to generate cheap debt—i.e., a debt value exceeding the Merton (1974) frictionless debt value. For example, at k=.30, and given other specified parameter values, α must exceed .0351 in order for $D_0 \geq D_0^M$. Otherwise, for α<.0351, debt costs (yields) exceed those obtained in a frictionless setting due to costs of financial distress, implying expensive debt.

Given positive costs of financial distress, debt becomes increasingly "less cheap" as leverage increases. In fact, for any given positive and finite alpha, the increasing marginal cost of debt due to k>0 implies that there will exist a B such that debt is no longer cheap in the sense that $D_0 \geq D_0^M$. I now formally derive marginal debt costs as a function of B, emphasizing a result whereby debt becomes infinitely expensive at the margin. The following lemma summarizes the result, which I refer to as the *choke condition*.

Figure 1 – Debt Value as a Function of Alpha and Costs of Financial Distress



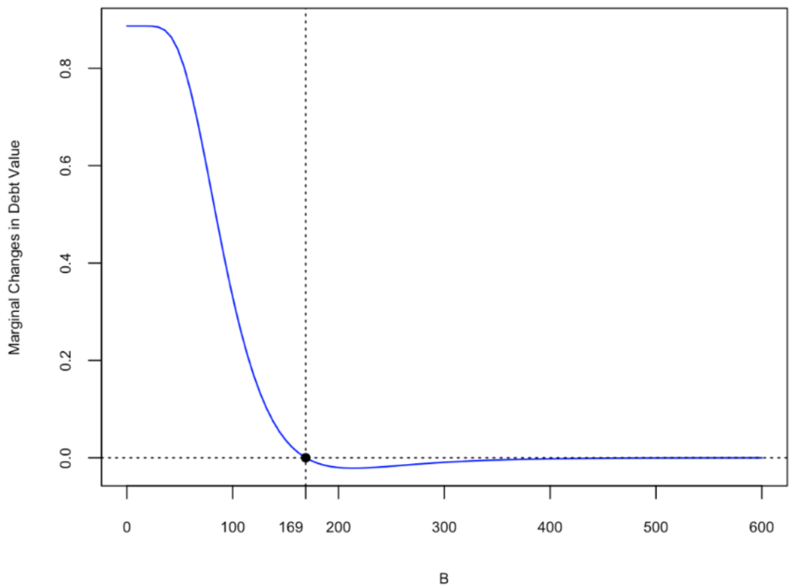
Notes: This figure plots debt value for combinations of k and α such that D_0 equals the frictionless debt value, D_0^M. Debt values to the right of the curved line are combinations of k and α such that D_0 is less than the frictionless debt value, while debt values to the left of the curved line are combinations of k and α such that D_0 is greater than the frictionless debt value. Parameter values are: V_0=100; B=80; r=.02 : μ=.10; α=.02; σ=.175; k=.30; T=6.0.

<u>Proposition 1 (The "Choke" Condition)</u>: $\frac{\partial D_0}{\partial B} = e^{-rT}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]$. For any k>0, there exists a unique B, B_k^*, which satisfies $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 0$. $\frac{\partial D_0}{\partial B} > 0$ and $\frac{\partial^2 D_0}{\partial B^2} < 0$ for k>0 and $B \in [0, B_k^*)$.

<u>Proof</u>: See Appendix A

When k=0, the standard comparative static result obtains in which debt value always increases as a function of B. However, for any k>0, there exists a finite leverage level, denoted as B_k^*, at which the comparative static switches signs to become negative. I label this crossing point the "choke condition" – the point at which the marginal cost of debt becomes infinite. Figure 2 displays how the key relation, $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$, varies as a function of B. At B=0, $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 1$. For k>0, $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$ is positive but decreasing up to the point at which $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 0$. After crossing the choke threshold, debt value begins to decrease with increases in B. A functional minimum will exist, after which the slope of the function turns positive and asymptotes to zero from below as B→∞.

Figure 2 – The "Choke" Condition



Notes: This figure displays $\frac{\partial D_0}{\partial B}$, i.e., marginal changes in debt value as it depends on *B*. $\frac{\partial D_0}{\partial B}$ crosses zero at the choke value, B_k^*, which in this case is $B_k^* = 169.0$. For *B*'s that exceed B_k^*, debt values decrease as *B* increases, implying that $B = B_k^*$ establishes an endogenous upper bound on the debt funding amount. Parameter values used to generate the figure are: V_0=100; r=.02: μ=.10; α=.02; σ=.175; k=.30; T=6.

The zero-crossing displayed in Figure 2 thus corresponds with an endogenously determined maximum debt value, implying a hard limit on debt issuance proceeds. Consequently, when there are costs of financial distress, with enough leverage, even when the cost of debt incorporates a substantial alpha that causes "cheap debt" at low to moderate leverage levels, debt eventually becomes infinitely expensive at the margin. This endogenous limit on debt funding provides a new and different rationalization as to why equity-constrained GPs finance themselves with outside equity capital through the LP-GP investment vehicle structure.[14,15]

Lastly, observe that my model is in certain respects reminiscent of Leland (1994). Both are tradeoff models constructed using continuous-time methods, and both incorporate costs of financial distress. Financial distress

[14] This is in contrast to Axelson, Stromberg and Weisbach (2009), who posit asymmetric information between GP's and LP's, with debt emerging as the low-cost source of funds with which to acquire fund assets.
[15] Given parameter values used to generate Figure 2, and considering three alternative alpha values of .00, .02 and .04, I specifically calculate the maximum achievable initial debt value, D_0^{Max}. Assuming an asset acquisition cost of V_0=100, the maximums are 72.39, 81.62 and 92.02, respectively. The first two maximum debt values produce debt-to-value ratios of 72.39% and 81.62%. This corresponds well with maximum LTV ratios observed in the commercial real estate mortgage market.

costs in both cases create endogenous bounds on debt proceeds – the point at which debt is infinitely expensive. My model is constructed for a closed-end PE fund, however, while Leland's model best applies to corporations as going concerns and perhaps open-end PE funds as well as hedge funds. Whereas debt in Leland's model is coupon-based with no maturity date, and with bankruptcy endogenously determined, debt is zero-coupon in my model with a finite maturity date matching the fund liquidation date. Bankruptcy happens only at debt maturity when fund asset value is less than what is owed on the debt. Most importantly, there is tax deductability of interest favoring debt financing in the Leland model. Taxes are not incorporated as an offset to costs of financial distress in my model, whereas the existence of GP alpha favors debt as a funding source.

IV. Step 2: The Compensation Contract and the GP's Leverage Choice Problem

The GP's compensation contract has two basic components: i) management fees that cover an assortment of activities necessary to finance, invest and operate the fund, and ii) an incentive fee that is paid at the end of the fund's life. Management fees come in a variety of shapes and sizes, including acquisition and monitoring fees (Metrick and Yasuda (2010)). In the case of PERE funds, there are also commonly property management, project management and development fees.

To simplify matters, I will assume that management fees are imposed by the GP to cover all of the estimated overhead costs. These costs, denoted as Φ^F, are included in the LP's capital that is contributed at the start of the fund life. They are to be drawn upon by the GP as various investment and operating costs are incurred. Management fees, once they are set by the GP, and included in contributed capital, do not impact the GP's fund leverage decision.[16] In contrast, the incentive fee, often referred to as carried interest, depends directly on the LP's prioritized preferred return on equity as well as the marginal cost of debt as they (may) vary as a function of fund leverage.

[16] If management fees were only paid as a constant fixed percentage of invested or contributed capital, and there were no other fees, one could argue that management fees affect fund capital structure choice. But as documented by Phalippou (2018) and others, the plethora of PE fees, such as transaction fees, monitoring fees, deal fees, and so on, many of which scale by total assets of the fund rather than contributed capital, suggest a management fee structure that is not dependent on relative fund leverage. The additional PERE fund fees noted above also scale by fund assets rather than fund equity. Finally, management fees, in theory and presumably in practice, are set to cover GP overhead and overhead only. Thus, overhead costs will either be invariant to or mostly depend on total fund size.

Although fund leverage is to a greater or lesser extent observable, and therefore potentially contractible, it remains true that fund leverage choice is controlled by the GP. Indeed, as Brown et al. (2021) show, the GP has several different debt funding levers available to it, where debt sources outside of the Op Co level can be hard for the LP to observe and control. If the GP is able to increase its expected carried interest payouts by increasing or decreasing fund leverage from that identified in the offering documents, one should expect to observe fund leverage "shading" by the GP in one direction or the other. In recognition of the potential for *ex post* opportunism, I hypothesize that the GP's fund leverage choice is directly managed through the carried interest contract. With this in mind I will now work out incentive compatible fund leverage choice (the carrot), and later introduce *ex ante* contracting on fund leverage levels (the stick) as a complement to determining fund leverage.

Let expected GP incentive fee revenue equal $\Phi^V(B;\psi,\rho)$, where $\psi \geq 0$ denotes the carried interest return hurdle rate and ρ, $0 \leq \rho \leq 1$, denotes the carried interest share percentage. Carried interest paid to the GP will be based on the time T liquidation value of the fund.[17] Specifically, carried interest is only paid at time T when total liquidation value of the fund's assets exceeds the time T priority claim payoffs of $B + (V_0 - D_0)e^{\psi T}$. Here, B is balloon debt payment due at time T, $V_0 - D_0$ is LP equity available for investment at the start of the fund's life (i.e., invested capital), and ψ is continuously compounded carried interest hurdle rate that establishes the LP's preferred return of capital.[18]

Let $\chi_0(B;\psi) = B + (V_0 - D_0)e^{\psi T}$. The quantity $\chi_0(B;\psi)$ is an exercise price above which carried interest is paid and below which incentive compensation is zero. The GP's resulting optimization problem can be stated as follows, where the GP chooses the fund capital structure to maximize expected incentive fees to be paid at the time of fund liquidation:

[17] According to its 2020 survey of management fees and terms study, *PREA* finds that 93% of PERE Value-Add and Opportunity fund survey respondents calculate incentive fees after the full return of capital calculated at the end of the fund's life.

[18] Phallipou (2021) notes that industry practice sometimes incorporates management fees that are part of the LP's contributed capital into the calculation of the carried interest hurdle value. Doing so effectively increases the hurdle, which in turn endogenously affects fund leverage determination and ultimately carried interest. I have considered this modification and found that it is straightforward to incorporate into the analysis, and that it does not change essential relations or magnitudes in meaningful ways. As a result, I will utilize investable capital rather than contributed capital in my baseline model when determining the preferred interest payment.

$$\underset{B}{Max}\ \Phi^V(B;\psi,\rho) = \mathbb{E}_0[Max\{0,\rho[\tilde{V}_T - \chi_0(B;\psi)]\}], \tag{2a}$$

$$\text{s.t. } \Phi^V(B;\psi,\rho) > \breve{\Phi}^V, D_0^{Min} \leq D_0 \leq D_0^{Max} \tag{2a'}$$

The constraints identified in equation (2a') are: i) Expected incentive fees exceeding a minimum, $\breve{\Phi}^V \geq 0$, to ensure participation of the fund manager, and ii) Endogenous fund leverage choice satisfying predetermined fund leverage bounds. These constraints can, among other things, reflect the GP's bargaining power and longer-run fundraising objectives. Throughout, I will assume a lower bound on $\breve{\Phi}^V$ of zero unless otherwise noted. Given that fixed fees cover overhead and other essential operating costs, the GP will never be willing to participate unless it expects to earn positive incentive fees. As for explicit limits on fund leverage, as noted above I put them aside for now to be taken up later in the paper.

Equation (2a) is recognized as a call option on fund payoffs that exceed those required to pay off priority claims, where the optimization problem can now be written as,

$$\underset{B}{Max}\ \Phi^V(B;\psi,\rho) = \rho\left[V_0 e^{(\mu+\alpha)T} N[h_1] - \chi_0 N[h_2]\right] \tag{2b}$$

$$h_1 = \frac{ln\left[{V_0}/{\chi_0}\right] + \left((\mu+\alpha) + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},\ h_2 = h_1 - \sigma\sqrt{T} \tag{2b'}$$

I am now in a position to solve the GP's variable compensation optimization problem, with Proposition 2 stating the result.

<u>Proposition 2 (Unconstrained Optimal Fund Leverage Choice)</u>: $\frac{\partial\Phi^V}{\partial B} = -\rho N[h_2]\left[1 - e^{(\psi-r)T}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]\right] = 0$ satisfies GP incentive compatible capital structure choice. For $\psi \geq r$, a finite incentive compatible debt value, B^*, exists in the range $B^* \in [0, B_k^*)$. B^* is also unique for any $k \geq 0$. Further, B^* does not depend on the carried interest percentage, ρ. Alternatively, for $\psi < r$, $B^* = 0$ is optimal.

<u>Proof</u>: See Appendix A

When $\psi \geq r$, inspection of the incentive compatibility condition reveals that there is a unique solution in which B^* is increasing in ψ. This follows from proposition 1, since $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$ is positive and strictly decreasing in B in the range $B \in [0, B_k^*)$. Note that a positive relation between the carry hurdle rate and leverage conforms with the previous empirical relation documented in section II.A.

The finding that the GP optimizes incentive fee payments with finite leverage, and with fund-level debt that is increasing in the carry hurdle rate, is not necessarily intuitive. To better understand the result, one can rearrange terms inside the bracketed term in proposition 2 as follows:

$$e^{\psi T} = \frac{1}{\frac{\partial D_0}{\partial B}} = \frac{e^{rT}}{N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}} \tag{3}$$

Given ψ, which appears only in the LHS of equation (3), preferred interest payments to the LP accrue in proportion to invested capital. The LHS of equation (3) measures the marginal benefit of reducing invested capital (increasing debt) over the life of the fund. Importantly, preferred interest does not vary as a function of fund leverage. The RHS of equation (3) quantifies marginal increases in the cost of debt as debt is substituted for equity in the fund's capital structure. With no fund leverage, the opportunity cost of debt capital is r (since the denominator is equal to one when B=0). As fund leverage increases above zero, so does the marginal cost of debt. The GP increases fund leverage until the marginal cost of debt over the life of the fund equals the marginal benefit of reducing the LP's equity footprint as it determines preferred interest payments. A higher ψ implies a higher marginal cost of equity as it applies to optimizing GP incentive fees, which in turn implies a higher marginal cost of debt in equilibrium, and thus higher fund leverage.

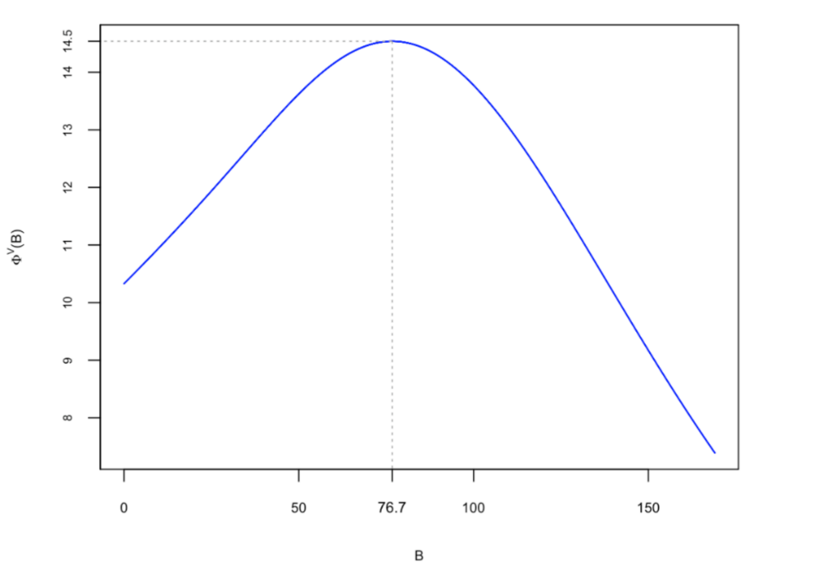
This tradeoff in the cost of debt versus equity stands in stark contrast to conventional relations in which the cost of equity capital increases in leverage. In PE, according to the standard carried interest compensation contract, the LP equityholder does not explicitly require any compensation for risk in its preferred rate of interest. This makes the LP equity capital particularly cheap at moderate to high fund leverage levels, which limits fund leverage. I note that this relation is revealing about LP attitudes towards risk. Granted, the LP does have an additional residual claim on fund profits, which it shares with the GP. But the pro rata sharing of residual profits makes that part of the capital stack irrelevant to the optimal fund leveraging decision made by the GP, which undercuts the commonly held notion that convex payoffs result in unbounded risk-taking.

In the absence of a minimum fund leverage constraint, equation (3) also clarifies why $B^* = 0$ when $\psi = r$. Furthermore, when $\psi < r$ there is no internal solution to the IC condition stated in proposition 2. In this case,

$B^* = 0$, since $\frac{\partial \Phi^V}{\partial B}$ is negative for all $B \in [0, B_k^*)$. The GP is incentivized to avoid fund leverage altogether, even though debt may be cheap due to positive alpha. It does so because the cost of equity capital with respect to the preferred interest rate is even less than the risk-free rate of interest.

Figure 3 provides a visual depiction of GP's fund leverage choice outcome, B^*, conditional on $\psi > r$. Incentive contract terms are ψ=.09, ρ=.20, which correspond with the median carried interest hurdle rate and carried interest share values found in my earlier empirical analysis utilizing the *Real Estate Alert* data. The figure shows that B^* equals 76.7, which generates expected carried interest of $\Phi^V = 14.54$. Time t=0 debt value, D_0^*, equals 63.09. Given that $V_0 = 100$, invested capital is 36.91. Endogenously determined incentive compatible fund leverage of 63.09% is right in between the two mean values of 62.37% and 63.96% found in my earlier empirical analysis as reported in Table 2.[19]

Figure 3 – GP Choice of Fund Leverage to Optimize Carried Interest Payments



Notes: This figure displays the GP's expected incentive fee as a function of *B*. A unique optimum exists, in this case at B=76.7, which corresponds with D_0=63.1. Parameter values used to generate this figure are: V_0=100; r=.02 : μ=.10; α=.02; σ=.175; k=.30; T=6; ψ=.09, ρ=.20.

[19] I will follow industry convention and express leverage ratios based on the book value of assets (V_0) rather than the market value of assets once acquired ($V_0 e^{\alpha T}$).

Proposition 2 is a separation result, in that only the preferred interest rate, ψ, and not the carried interest share value, ρ, is required to determine incentive compatible fund leverage. Prior empirical analysis documented variation in the carried interest hurdle rate, but almost no variation in the carried interest share value centered at 20 percent. This is consistent with my model, where only the carried interest hurdle value, ψ, is necessary to nail down fund leverage.

As a consequence, the carried interest percentage parameter, ρ, remains free to ensure the GP's fund participation constraint is satisfied (see equation (2a')). The fact that this parameter is essentially invariant in practice suggests that: i) the participation constraint is never really binding at ρ=.20, implying that GP's are effective at preventing LP's from reducing ρ to the point where the participation constraint becomes binding, or ii) ρ=.20 assumes the role of an industry convention, essentially a universal constant of PE nature, with the GP addressing participation in other ways. I will circle back to this issue later in the paper when assessing incentive contracting when LP's optimally target net-of-fee fund returns.

The following result provides stronger justification of my model as it relates to the assumed fundraising structure.

Corollary 1 to Proposition 2: Φ^V is homogeneous of degree one in V_0 and B.

Proof: Follows by inspection of Φ^V as stated in equation (2b) and recalling that D_0 is also homogeneous in V_0 and B.

This result implies that the GP can simultaneously satisfy fund leverage and the target LP equity contribution by scaling total fund size up or down as necessary. In other words, given linear homogeneity of Φ^V in V_0 and B, an approach of holding V_0 constant when determining the fund's optimal capital structure does not impose unreasonable restrictions on the PE fundraising process.

Intuition might suggest that a variable fund size that is scalable through the amount of equity raised can counteract the equity dilution effect associated with increased leverage. However, once the equity raise is realized, and given that fund asset characteristics (including alpha) do not change, the GP is subject to exactly the same leveraging incentives identified above given that the GP simultaneously optimizes fund leverage structure and the contributed equity raise. In fact, an approach of first identifying fund size and then the fund's capital structure, and therefore its required equity contribution, is consistent with industry practice, in that it also preserves the constant alpha value restriction. This approach is moreover consistent with the model and findings of Axelson et al. (2009). In their model total fund size is set exogenously, with the split between debt and equity, as well as fundraising sequencing, determined endogenously. They show that a fundraising sequence of first raising equity capital at the fund level, and then applying leverage at the asset level at the time of investment, serves as a commitment device that optimizes expected GP fees.

An alternative sequencing approach of first raising LP equity (setting $E_0 = \bar{E}_0$) and then determining an optimal fund asset size, V_0, as it depends on total leverage introduces additional complexities. As shown in Appendix B, in this case debt value is self-referencing due to the fact total fund size depends endogenously on debt quantity. A finite optimal debt level nonetheless exists using this approach, which I denote as $\breve{D}_0$, where a different tradeoff ensues. In particular, by totaling differentiating $\breve{D}_0$ with respect to B, in equilibrium I find that

$$\frac{1}{\frac{\partial \breve{D}_0}{\partial B}} = \frac{N[\breve{h}_1]e^{(\mu+\alpha)T}}{N[\breve{h}_2]} \tag{4}$$

where $\breve{D}_0$ denotes time t=0 debt value when fund size itself depends on D_0, and where $\breve{h}_1$ and $\breve{h}_2$ depend on $\breve{D}_0$ (see appendix B).

Here, with preferred interest paid to the LP being invariant due to fixing the committed equity amount, the GP adds leverage until marginal debt costs (LHS of equation (4)) slightly exceed the marginal benefits of acquiring additional assets, $e^{(\mu+\alpha)T}$.[20] This formulation makes explicit the reliance on constant alpha as a function of scale, which is dubious as the literature has shown (see, e.g., Kaplan and Lerner (2010) and Lopes-de-Silanes, Phalippou and Gottschalg (2015), among others). I further show that fund leverage in this alternative model structure exceeds fund leverage in my baseline model, in the sense that marginal debt costs are higher in equilibrium whenever $\psi \leq \mu+\alpha$. This is intuitive, since ψ measures GP benefits of increasing leverage given a

[20] "Slightly exceed" follows from the fact that $\frac{N[\breve{h}_1]}{N[\breve{h}_2]} \geq 1$.

fixed fund size, while $\mu+\alpha$ reflects benefits of increasing fund size through leverage when holding contributed equity constant.

Thus, in a relatively unrestricted setting, given a convex carried interest payoff function I show that there are limits to GP risk-seeking through fund leverage ratcheting. This result obtains even when there are no costs of financial distress (k=0) and for managers that may or may not be skilled ($\alpha \geq 0$). The tradeoff in this case is entirely due to the increasing marginal costs of debt due to fund expansion versus the (nearly constant) marginal benefits to acquiring additional assets and holding them over the life of the fund. As in the baseline fundraising case, the constant cost of preferred equity as a function of fund leverage plays a critical (but in this case silent) role, thus helping explain the limited fund size result.

The bracketed term in proposition 2 indicates a direct mapping between the carried interest return hurdle, ψ, and the leverage measure, B^*. This relation can be restated by isolating the carried interest hurdle rate as a function of debt value parameters, which can then be used to characterize comparative static relations that exist between B^* and relevant parameter values. The following corollary states the results.

<u>Corollary 2 to Proposition 2</u>: Given $\psi \geq r$ and for the relevant range, $B^* \in [0, B_k^*)$, satisfying incentive compatibility implies that $\psi = r - \frac{1}{T} ln\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]$. With this, the following comparative static relations obtain: $\frac{\partial B^*}{\partial \psi} > 0$; $\frac{\partial B^*}{\partial k} < 0$; $\frac{\partial B^*}{\partial \alpha} > 0$; $\frac{\partial B^*}{\partial \sigma} \lesseqgtr 0$; $\frac{\partial B^*}{\partial T} \lesseqgtr 0$ $\frac{\partial B^*}{\partial r} \lesseqgtr 0$.

<u>Proof</u>: See Appendix A

$\frac{\partial B^*}{\partial \psi} > 0$ and $\frac{\partial B^*}{\partial k} < 0$ were discussed previously. Although the positive comparative static relation between GP skill (α) and fund leverage is intuitive in the context of the model, it provides an interesting contrast to predictions that focus on different agency frictions that exist between GP's and LP's. A focus on positive financial and governance engineering effects in the spirit of Jensen (1989) suggest a positive relation between leverage and alpha, with causation going from the former to the latter. More recent empirical findings of Demiroglu and James (2010) and Axelson et al. (2014) emphasize agency costs of debt as it affects fund performance, with similar causation but a negative relation. Brown et al. (2020) document a positive empirical relation between alpha (performance) and leverage, but the causal direction is unclear. My model indicates a positive relation, but with causation going from alpha to fund leverage. This relation is consistent with Andrade and Kaplan's (1998) findings that operationally efficient PE funds have lower overall distress costs, resulting in a better ability to bear leverage risk. In my model, for similar reasons, high alpha GP's can assume greater leverage, and in fact do so in equilibrium because operational efficiencies offset costs of financial distress.

In general, leverage can increase or decrease in the fund's asset risk (σ). For realistic parameter values, the relation is negative. This follows because greater asset risk increases the marginal debt cost relative to the constant marginal cost of equity capital, which reduces the GP's demand for debt. A negative relation between fund leverage and asset risk is consistent with the recent findings of Brown et al. (2020).

Although ambiguous in general, for realistic parameter values a positive relation between fund life, T, and fund leverage obtains. A longer fund life favors the persistent effects of an upward drift in asset values relative to random price variation that can dominate over shorter horizons. This increases the fund's ability to bear leverage. Lastly, an increase in interest rate (r) has a non-monotonic effect on fund leverage.

V. Step 3: LP Performance Measurement, Parameter Selection and Model Calibration

V.A. Gross- and Net-of-Fee Returns

Gompers, Kaplan and Mukharlyamov (2016) document that institutional investors "focus more on absolute performance [typically measured by IRR] as opposed to risk-adjusted return," while Korteweg (2019) states there is no consensus in PE as to how risk should relate to return, where "this lack of agreement has likely contributed to the lack of formal quantitative risk adjustment in practice."

Following industry practice, in the model LP performance is measured by IRR. The gross-of-fee IRR is simply,

$$\lambda^G = \frac{1}{T}\left[ln\left[\frac{\mathcal{E}_T(B)}{E_0(B)}\right]\right] \tag{5}$$

where $\mathcal{E}_T(B)$ is the expected value of invested capital evaluated at time T and prior to the payment of incentive fees and $E_0(B) = V_0 - D_0(B)$ is equity contributed by the LP for investment and prior to payment of management fees.

The time T expected value of equity is a call option on the assets of the fund after debt repayment, expressed as follows:

$$\mathcal{E}_T(B) = \int_B^\infty [\tilde{V}_T - B] f(\tilde{V}_T | V_0) d\tilde{V}_T = V_0 e^{(\mu+\alpha)T} N[\hat{d}_1] - BN[\hat{d}_2] \tag{6}$$

with $\hat{d}_1 = \frac{ln[V_0/B] + ((\mu+\alpha)+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T}$.

Expected net-of-fee returns will depend on incentive fee payments as well as fixed management fees that are contributed as part of committed capital. The fundamental relation that describes net-of-fee investment performance is as follows:

$$e^{\lambda^N T}[E_0(B) + \Phi^F] = \mathcal{E}_T(B) - \Phi^V(B) \tag{7}$$

where λ^N is the LP's expected net-of-fees holding period return (IRR); $E_0(B) + \Phi^F$ is committed LP capital at t=0, with total management fees denoted by Φ^F; and $\Phi^V(B)$ is the expected incentive fee to be paid at time T to the GP as carried interest. Solving for λ^N in equation (7) results in:

$$\lambda^N = \frac{1}{T}\left[ln\left[\frac{\mathcal{E}_T(B) - \Phi^V(B)}{E_0(B) + \Phi^F}\right]\right] \tag{8}$$

The properties of the expected holding period return will vary depending on parameter selection. Putting aside fund leverage constraints and the GP's incentive compatible debt choice for the moment, in certain cases when financial distress costs, k, are small relative to α, infinite expected returns can be achieved at finite leverage levels. This outcome can be seen in the denominator term of equation (8), where it is possible for $E_0(B) + \Phi^F$ to approach zero from above as B increases. For example, given k=0, $\alpha = .02, \Phi^F = 5$, and given other parameter values used to generate Figure 4, at B=172.7 (D_0=105) the expected holding period return blows up to become infinite. Alternatively, when k=.3 a "break-even" alpha obtains, with α=.062 such that D_0=105 at $B = B_k^*$ =217.4. At this point the expected holding period return goes to infinity. Given k=.3, for alpha values in excess of .062 the expected holding period return blows up at even lower B values.

The ultimate cause of this phenomenon is an arbitrage. Fund assets are acquired at a cost of V_0, where asset value in the hands of the GP goes immediately to $V_0 e^{\alpha T}$. For small to negligible financial distress costs, debt value at t=0 can exceed V_0 for B large enough, as lenders too recognize the value-enhancing ability of the GP. In the case of k equal to zero, any positive alpha is sufficient for there to exist a finite B such that "cheap" debt proceeds equal the acquisition value of fund assets. Such outcomes do, however, contradict rationales that rely on GP financial constraints to explain the existence of the PE fund model, since the GP can finance the entire fund asset acquisition cost itself with the use of "cheap" debt. In my model, which posits positive and not insignificant costs of financial distress that result in endogenously determined maximum debt levels, such pathological outcomes occur only rarely when realistic parameter values are imposed.

V.B. Parameter Selection and Model Calibration

The purpose of this sub-section is to carefully analyze the model's ability to explain the data. In particular, I hypothesize that a meaningful link between the PE incentive contract and GP fund leverage choice exists, with outcomes that also explain observed LP target returns. At this point I am taking the incentive contract as given. In the next section I endogenize LP return targeting in an attempt to shed additional light on PE incentive contracting practices.

As previously summarized, in my analysis of PERE fund data I find that fund leverage ranges between 50.0% and 75.0%, with a central tendency in the range of 60.0% to 70.0%. Net-of-fee target-IRR's generally range between 13.0% and 20.0%, with fee drag in the range of 3.5% to 4.0%. These are the critical values I want to explain through parameter selection and model calibration. I will also address the positive empirical relation I found earlier between fund leverage and fund performance.

Parameter selection will start with management and incentive fee contract terms, and then move into fund asset characteristics and debt contracting variables.

Management Fees: As discussed in some detail earlier, proper accounting for management fees is more complex than typically characterized. Many management fees, such as acquisition, monitoring and development fees, scale directly to total fund size as opposed to invested capital. Moreover, management fees are paid over the entire life of the fund, and not simply over the cash-based duration of fund investment. For example, although fund durations may commonly be on the order of five to seven years, management fees are incurred continually over a typical fund life of say 10 years. Consequently, I will assume 1.50% in total management fees per annum (as displayed in column (1) of Table 1), taken over a 10-year fund life, to result in 15.0% total management fees as a percentage invested capital. Then I rescale these fees based on representative fund leverage ratios, resulting in total fixed management fees equal to 5.0% of the fund's asset acquisition cost.

Incentive Fee Contract Variables: As previously documented, the carried interest share is commonly set at 20.0% (see column (3) of Table 1) with almost no variation around the mean-median value. The carry hurdle rate is centered at 9.0%, with most observations in the 8.0 to 10.0% range. For baseline model estimation purposes, in this sub-section I will specify a carried interest share of 20.0% and a carried interest hurdle rate of 9.0%, with no catch-up fee provision term. Catch-up fees will be considered in the next section of the paper.

Fund Assets' Unlevered Equilibrium Rate of Return: PERE funds invest in commercial real estate assets. Reference to monthly reports issued by *Green Street*, which produces regular estimates of expected returns to the industry and particular property types, suggests unlevered returns of around 6.0%. *Real Estate Research Corporation (RERC)* also produces estimates of unlevered discount rates for major property types. As of Q4 2020, their estimates range from 5.0% to 8.5% for A+ to A quality property.

Value-add and opportunity PERE funds typically acquire assets that require repositioning, and oftentimes significant development or redevelopment. This increases asset risk relative to otherwise equivalent income-producing assets. According to *RERC*, unlevered discount rates for B and C quality assets range from approximately 7.0% to 12.0%. In referencing Pagliari's (2020) analysis of value-add and opportunity funds, he pegs unlevered expected asset returns at approximately 10.0%. I am not aware of any other direct evidence on the topic, so will designate μ=.10 as my base-case value.

Fund Assets' Unlevered Standard Deviation of Return: Empirical estimates of PERE fund return volatility exist in industry publications (e.g., *CEM Benchmarking 2020*). But such estimates are generally made on a portfolio or index of levered funds on a net-of-fee basis. With that in mind, fund volatility estimates are typically in the 17.5% to 25.0% range. Pagliari (2020) generates direct volatility estimates on value-add and opportunity PERE funds in the 15.0% to 20.0% range. Based on my own analysis of *Preqin* data, I find the standard deviation of IRRs realized on 399 value-add and opportunity PERE funds to be in the 16 to 17 percent range. Altogether, these data points result in σ=.175 as my base-case value.

I note that the base-case fund volatility is significantly below asset volatilities estimated or assumed in VC and buyout funds (e.g., Cochrane (2005), Metrick and Yasuda (2010), Sorensen, Wang and Yang (2014)). But, given the highly specific idiosyncratic risks associated with these investments, they generally have a low correlation structure that significantly reduces variation in payoffs at the fund level. Resulting variation in fund-level payoffs is therefore much more in line with my PERE-based fund payoff volatility estimate.[21] Furthermore, PERE funds often specialize by property type and geographical area or region, resulting in a strong correlation structure relative to that observed in buyout and VC funds.

Alpha: Alpha is estimated on a gross-of-fee basis. There are no direct estimates of gross-of-fee value-add and opportunity PERE fund alphas that I know of. As noted earlier, there are good reasons to believe that operational, governance and financial engineering benefits to PERE are limited relative to benefits available in PE. Limited benefits are closely related to the existence of a viable and liquid parallel public market for the ownership of commercial real estate. That said, the repositioning and redevelopment of assets held PERE funds do offer opportunities to add value relative to holding run-of-the-mill income-producing property.

Two recent studies implicitly document PERE fund alpha's on a net-of-fee basis. Gupta and Van Nieuwerburgh (2021) estimate that closed-end PERE funds lose 17 cents on average for every dollar invested. Given fund durations of five to seven years on average, this equates to approximately a 2.0% to 3.0% negative net-of-fee alpha. Applying a standard mean-variance framework, Pagliari (2020) and Bollinger and Pagliari (2019) generate similar net-of-fee alpha estimates for PERE value-add and opportunity funds. Both papers make risk

[21] For example, Sorensen et al. (2014) assume 60% volatility at the deal-level, with 20% pairwise correlations and 15 deals in a fund. This reduces volatility to 25% at the fund level.

adjustments and do not explicitly account for liquidity differences between PERE funds and the liquid benchmark indices.

The issue of whether institutional PERE LPs, particularly pension funds which represent approximately 50% of all investment in PERE value-add and opportunity funds, demand a standard asset risk premium, and how much it might be, is an unsettled issue.[22] So is the issue of illiquidity risk and fund pricing. There are some (Asness (2019), Riddiough (2021)) that have noted apparent institutional preferences for illiquid PE investment vehicles that veil price volatility of PE funds relative to liquid and more visibly volatile public market alternatives. The argument is that a 2%-3% negative net-of-fee PERE return is the price that institutional investors are willing to pay avoid measured volatility.

Altogether, this leads me to choose a base case α=.02. Given the unsettled state of PERE asset pricing, this estimate seems reasonable based on observed net-of-fee performance ranging from 0.0% to −3.0 %, with observed fee drag in the 3.5%-4.0% range. Going forward, with α=.02 as the base case, we will examine alphas in fairly tight range of up to four percent.

Fund Duration and Debt Term: I am in possession of *Preqin* data on 78 PERE Value-Add and Opportunity funds that have liquidated and for which I have a full set of cash flows. In these data the average fund life is 10.5 years. Durations and weighted average fund lives are significantly shorter, however. These data generate an average fund duration of 4.7 years (as measured by the method suggested in Phalippou and Gottschalg (2009)) and an average weighted average life of just under 5.0 years. Other data I have seen indicate weighted average PERE fund lives in the four to eight year range. Given these data points, I will take T=6 years as my base-case value.

Risk-free Rate of Return: The Treasury rate is often referenced as the risk-free rate. Recent research suggests that the risk-free rate exceeds the Treasury rate due to a convenience yield. In recognition of this and the fact

[22] See Riddiough (2021) for detail on the PERE pension fund ownership share. On the risk-return relation, as previously discussed, see Gompers, Kaplan and Mukharlyamov (2016) and Korteweg (2019). Phalippou (2021) adds, "The PE industry is less of a puzzle, however, if one recognizes the multiple layers of ageny conflicts and the complexity of measuring risk and returns of illiquid assets, whose effects are exacerbated by the lack of knowledge on that particular issue by some of the decision makers."

that for the last ten-plus years we have been living in a particularly low interest rate environment, the risk-free rate is specified as r=.02.

Costs of Financial Distress: Financial distress costs were previously pegged at k=.30. As discussed earlier, this estimate is higher than the 10.0% to 20.0% costs estimated by Andrade and Kaplan (1998) with buyout funds, and is at the high end of the range estimated by Brown et al. (2006) in their analysis of commercial mortgage loan distress costs. The high-end cost estimate is justified based on the higher risks associated with the types of assets held in Value-Add and Opportunity PERE funds.

Table 4 displays the calibrated model results. There are six reported statistics of interest, which are identified across the top of the table. Based on assets that are acquired by the fund at time t=0 for a cost of V_0=100, the first statistic is the time t=0 debt value, D_0. Debt quantity is determined endogenously by the GP in order to maximize expected incentive fee payments (see Proposition 2). In the base case, D_0=63.09. This value compares to the mean fund leverage of 62.37% and 63.96% found in the PERE data. For alternative parameter values reported in the left-hand column of Table 4, equilibrium debt values as a percentage of asset acquisition cost are tightly clustered in the 58% to 68% range.

The second statistic is the expected incentive fee payment, Φ^V (see column (2) of Table 4). This value ranges from 12.25 to 17.09 in the model calibration, with the base case value of 14.54. In the base case the LP is expected to receive all of the first 63.34 in profits after debt repayment as preferred interest when available (36.91 in invested capital earning 9.0% compounded continuously over a six-year fund life).

To put total fees into perspective, the third statistic displayed in column (3) shows expected total fee payments as a proportion of the expected terminal fund equity, $\mathcal{E}_T$ (expected terminal fund value less the debt payoff amount). This percentage is seen to vary within a tight range of 14 to 16 percent. Twenty-five to 30 percent of all fees are management fees, with the remainder attributable to carried interest. These proportions are in line with estimates provided by Phalippou (2021) in his analysis of PE fee payouts. Carried interest as a percentage of terminal fund enterprise value is, however, somewhat on the low side as compared to Phalippou's findings. His estimates include accounting for catch-up fees, however, which I have yet to address.

Table 4 – Baseline Calibrated Model Outcomes

Parameter Set	Initial Debt Value D_0 (1)	Incentive Fee Φ^V (2)	~ Total Fees $\Phi^T/\mathcal{E}_T$ (3)	Gross T-IRR λ^G (4)	Net T-IRR λ^N (5)	Fee Drag $\lambda^G - \lambda^N$ (6)
Base Case	63.09	14.54	.1515	.2085	.1674	.0411
k=.20	67.18	14.81	.1600	.2214	.1765	.0449
k=.40	59.91	14.33	.1454	.1998	.1612	.0386
α=.01	59.42	12.25	.1420	.1827	.1456	.0371
α=.03	67.03	17.09	.1614	.2372	.1915	.0457
σ=.15	67.66	14.54	.1574	.2242	.1795	.0447
σ=.20	58.79	14.71	.1475	.1960	.1575	.0385
r=.01	64.78	15.23	.1550	.2184	.1756	.0428
r=.03	61.17	13.85	.1477	.1983	.1590	.0393
Ψ=.075	60.10	15.41	.1526	.2016	.1616	.0400
Ψ=.105	65.55	13.69	.1500	.2143	.1723	.0420

Notes: Parameter values resulting from the empirical literature and my own estimates are: V_0=100; r=.02 : μ=.10; α=.02; σ=.175; k=.30; T=6; ψ=.09, ρ=.20.

The fourth, fifth and sixth statistics as reported in columns (4)-(6) of Table 3 are targeted gross-of-fee fund performance, λ^G, targeted net-of-fee fund performance, λ^N, and fee drag, $\lambda^G - \lambda^N$. Modeled net-of-fee performance matches well with empirically documented target returns, ranging from 14.56% to 19.155% in the model calibration estimates. These target returns nest within the previously documented empirical range of 13.0% to 20.0% Calibrations further indicate fee drag in the 3.71% to 4.47% range, which is only slightly to the high side relative to the 3.5% to 4.0% range documented in Table 2.

Note the negative relations between fund leverage and target return as they depend on asset price volatility and the risk-free rate. The asset volatility relation confirms prior comparative static results based on realistic parameter value combinations. The underlying intuition for the volatility result is that increases in fund asset volatility increase the marginal cost of debt relative to the fixed preferred equity interest rate, to reduce fund leverage. Similarly, an increase in the risk-free rate reduces fund leverage due to the discounting effect, which also decreases LP returns.

Given endogenously determined fund leverage, I can assess empirically documented positive relations between the carry hurdle rate and target-IRR, fund leverage and target-IRR, and the resulting implied positive relation between the carry hurdle rate and fund leverage. These relations are all consistent with correlation results reported previously. The key to understanding these relations is to first note that, due to the tradeoff between marginal debt and preferred equity costs, endogenously determined fund leverage increases in the carry hurdle rate. A higher carry hurdle rate along with higher fund leverage combine to increase targeted net-of-fee returns.

In summary, results reported in Table 4 indicates that my model successfully explains observed PERE fund leverage, target LP returns, the effects of fee drag on fund performance, and positive relations between the carry hurdle rate, fund leverage and performance. Causation in the model throughout goes from the incentive fee contract and fund characteristics to fund leverage and finally to fund performance.

VI. Step 4: LP Return Targeting, GP Skill Heterogeneity, and Catch-Up Provisions

VI.A. LP Return Targeting

In this section I consider the question of *why* the PE market has settled on a two-part incentive contract with nearly invariant parameter values as an industry standard. Doing so will require me to take a stand on what exactly LPs are optimizing. But as a first step, it is useful to know *who* the LPs are under consideration? Riddiough (2021) documents that, over the past 20 years, pension funds have had a 50 to 70 percent investor share of all value-add and opportunity PERE funds. Thus, given that pension funds are the dominant investors in these funds, I next ask, what does the pension fund investment objective function look like.

As previously highlighted, the literature has clearly established that pension funds are not prototypical expected (concave) utility maximizers that engage in a granular examination of risk-return tradeoffs. Rather, numerous studies have documented that pension funds focus primarily on performance as measured by IRR, with only coarse reference to risk. Axelson et al. (2009) provide the following lead-off quote expressing the GP's perspective: "Practitioner: Ah yes, the M-M theorem. I learned about that in business school. We don't think that way at our firm. Our philosophy is to lever our deals as much as we can, to give the highest returns to our LPs." Gompers et al. (2016) affirm this approach by documenting that, "limited partners focus more on absolute performance as opposed to risk-adjusted returns" and that "[buyout] investors target a 22% IRR on their investments on average." Korteweg (2019) sums up the state of our knowledge on the risk-return tradeoffs in PE: "This lack of agreement [on how to estimate risk and return in PE or on relevant performance benchmarks] has likely contributed to the lack of formal quantitative risk adjustment in practice."[23]

With these general findings in mind, and consistent with earlier empirical evidence on expected PERE fund performance, I will assume that LPs optimize through a process of *return targeting.* To produce the return target, the LP poses the following question: Given a hypothetical equivalent fund that generates zero alpha with no associated fees, what is the maximum return available when optimizing with respect to fund leverage? This exercise not only produces a unique return target, which I denote as λ^*, but also a *fund leverage target*. Denoted by $\overline{D}_0$, the fund leverage target establishes an observable upper bound on fund leverage that the LP is willing to accept, subject to meeting the return target. In other words, return targeting generates the following two constraints that must be simultaneously met for the LP to invest: $\lambda^N \geq \lambda^*$ and $D_0 \leq \overline{D}_0$. When both constraints bind, the LP expects to break even, making a zero risk-adjusted return. When there is slack in one or both of the constraints, the LP expects a positive risk-adjusted return.

To determine the target return, the LP optimizes equation (8) with respect to B, given that $\alpha = \Phi^F = \Phi^V = 0$. The following lemma summarizes the result.

<u>Lemma to Proposition 3</u>: Given $\mu > r$, $k > 0$, and $\alpha = \Phi^F = \Phi^V = 0$, an internal return target optimum exists with $B = \overline{B}$ that satisfies $\frac{E_0}{\mathcal{E}_T} = \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$.

<u>Proof</u>: The FOC follows directly from equation (8). Existence follows from the fact that $E_0, \mathcal{E}_T, \frac{-\partial E_0}{\partial B}, \frac{-\partial \mathcal{E}_T}{\partial B}$ all exceed zero for $B \in [0, B_k^*)$. At B=0, the RHS of the FOC above exceeds the LHS, while the LHS exceeds the RHS for B sufficiently close to B_k^*. These relations along with continuity imply at least one internal zero crossing point.

[23] For additional supporting evidence, see, among others, Axelson et al. (2014), Lerner et al. (2009), Boyer et al. (2021), Phalippou (2021), Andonov et al. (2017), Andonov and Rauh (2017), and Bodnaruk and Simonov (2016). I would also note the constant carry hurdle, which indicates a lack of risk adjustment in the cost of LP equity as a function of fund leverage, is revealing about LP risk attitudes.

I note that uniqueness is not guaranteed in general. As a consequence, I find an internal optimum by applying the FOC and then verify that it is a maximum. For any of the realistic parameter values used to calibrate the model in this study, I find only one internal optimum. The optimum follows from the fact that the RHS of the FOC starts at a higher initial value at B=0, but declines at a faster rate in B than does the LHS of the FOC.

Recall that, after matching first moments, model parameters of V_0=100; r=.02 ; μ=.10; σ=.175; k=.30; T=6, were used in the base-case analysis. With these parameter values, optimized LP return and leverage targets as described above result in $\lambda^* = .1672$ and $\overline{D}_0 = 64.95$. These calibrated values are remarkably consistent with target return and fund leverage outcomes previously documented in the PERE data. From Table 1 the PERE data show a mean target IRR of 16.71%, which is essentially equivalent to the calibrated return target of 16.72%. From Tables 2 and 3, median as well as modal fund leverage equals 65.0%, with 131 of 364 observations (36.0%) clustering at that value. This is also essentially equivalent to the calibrated value of 64.95%.

The LP's optimized return target and fund leverage constraints are now incorporated into baseline PE incentive contracting by considering the GP's constrained optimization problem. Here the GP, which is endowed with positive alpha, extracts fees to the point where, subject to not exceeding the fund leverage target, the target return constraint is binding. In solving the optimization problem the carry hurdle rate, ψ, is specified exogenously to fall within an empirically justifiable range. The carried interest share, ρ, as well as fund leverage, B, are determined endogenously. Once the optimal ρ and B are identified, the GP commits to those values in the offering documents.

Formally, the GP's optimization problem can be written as follows:

$$\underset{\rho,B}{Max}\, \Phi^V(\rho, B; \psi) = \rho\left[V_0 e^{(\mu+\alpha)T} N[h_1] - \chi_0 N[h_2]\right] \tag{9}$$

$$\text{s.t. } D_0(B) \leq \overline{D}_0,\ \lambda^N(B,\rho) \geq \lambda^*,\ 0 < \rho < 1 \tag{9a}$$

Note that equation (9) differs from equation (2b) only in that (9) it is a joint maximization problem involving both B and ρ.

Prior to solving this problem, I make three observations. First, reference to equations (7) and (8) indicate that incentive fees are strictly decreasing in λ. Second, $\Phi^V(\rho, B; \psi)$ is strictly increasing in ρ (see equation (9) above), with the interest carry share acting as a linear scaling factor. Third, based on these first two facts, and by combining equations (7) and (9), the GP is able to extract fees to the point where the target return constraint binds. Specifically, the GP sets $\bar{\rho}$ such that,

$$\bar{\rho} = \frac{\mathcal{E}_T - e^{\lambda^* T}[E_0 + \Phi^F]}{V_0 e^{(\mu+\alpha)T} N[h_1] - \chi_0 N[h_2]} \tag{10}$$

Two additional observations are in order. First, for $\psi < \lambda^*$, the target return constraint cannot be satisfied when $\rho=1$. That is, at least some profit sharing by the GP through interest carry must occur. Second, GP's will exit the market when they cannot meet the return target constraint with a positive carried interest share. That is, GP's will not contract for $\rho \leq 0$). As a result, based on these structural considerations, the $0 < \rho < 1$ constraint will always be satisfied and hence can be ignored.

To solve (9) subject to (9a) I form the Lagrangian, with the resulting KKT conditions:

$$\mathcal{L}(B, \rho) = -\Phi^V(\rho, B; \psi) + \mu_1\big(\lambda^* - \lambda^N(B, \rho)\big) + \mu_2(D_0(B) - \overline{D}_0) \tag{11}$$

where $\mu_1, \mu_2 \geq 0$ denote Lagrange multipliers. With this I am now in a position to state the major result of this subsection.

<u>Proposition 3 (The Constrained Baseline Contract When LPs Target Returns)</u>: When maximizing incentive fees the target return constraint always binds, with $\mu_1 = T[\mathcal{E}_T - \Phi^V] > 0$. As a result, the GP sets the carry interest share according to equation (10) in order to just meet the target return constraint. In general, depending on parameter values, the target fund leverage constraint may or may not be binding. When the fund leverage constraint binds, $\mu_2 = \frac{1}{\frac{\partial D_0}{\partial B}}\left[\frac{\partial \mathcal{E}_T}{\partial B} - \frac{\partial E_0}{\partial B}\left(\frac{\mathcal{E}_T - \Phi^V}{E_0 + \Phi^F}\right)\right] > 0$. Target return and fund leverage constraints simultaneously bind if and only if $\frac{E_0 + \Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$. This inequality relation is independent of ψ.

<u>Proof</u>: See Appendix A.

Because incentive fees are strictly decreasing in λ, and because GP incentive fees scale linearly in the carried interest share, ρ, it is optimal for the GP to solve the optimization problem as a two-part tariff. It first determines the optimal fund leverage, and then given the optimal $\bar{B}$ the GP finds $\bar{\rho}$ according to equation (10).

In general, the target fund leverage constraint may or may not be binding, depending on parameter values. It will not be binding when μ_2=0 for some B such that $D_0(B) < \bar{D}_0$. Given the calibrated parameter values used in my analysis, I find that the target fund leverage constraint always binds, implying that μ_2>0 for $\bar{B}$ such that $D_0(\bar{B}) = \bar{D}_0$. Thus the LP essentially breaks even from a risk-return perspective. Interestingly, μ_2, the shadow value associated with relaxing the fund leverage constraint, equals the marginal cost of debt, $\frac{1}{\frac{\partial D_0}{\partial B}}$, multiplied by the (partially adjusted) increase in LP return associated with a marginal dollar of debt, $\left[\frac{\partial \mathcal{E}_T}{\partial B} - \frac{\partial E_0}{\partial B}\left(\frac{\mathcal{E}_T - \Phi^V}{E_0 + \Phi^F}\right)\right]$.

One can simply use the inequality relation, $\frac{E_0+\Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$, to verify that the GP participates in the fund with $\bar{\rho} > 0$ and with the fund leverage constraint binding at $D_0(\bar{B}) = \bar{D}_0$. The participation constraint, $\frac{E_0+\Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T}$, requires finding an $\alpha^{Min} > 0$ such that $\bar{\rho} = 0$. Then for any $\alpha > \alpha^{Min}$ the GP participates and sets $\bar{\rho}$ according to equation (10), with a net-of-fee return to the LP of $\lambda = \lambda^*$. For the fund leverage constraint to bind, it must be that $e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$, which follows from the second KKT condition. In all cases, due to separation, the inequality relations do not depend on the carry hurdle rate, ψ.

Calibrated base-case parameter values used previously are V_0=100; r=.02 : μ=.10; σ=.175; k=.30; T=6. With these values I previously found that $\bar{D}_0 = 64.95$ and $\lambda^* = .1672$. GP participation is satisfied with α sufficiently large so that ρ>0. According to the base-case numerical analysis, α^{Min}=.0088 to produce $\frac{E_0+\Phi^F}{\mathcal{E}_T} = e^{-\lambda^* T}$. At that point I verify that $e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$.

The optimal fund structure as described is *ex ante*. There may, however, be *ex post* incentives for the GP to act opportunistically by deviating from the targeted fund leverage. This occurs when the incentive compatible fund leverage from the unconstrained problem (see step 2 in section III) is less than the target.[24] The ability of the GP to act on its incentives will depend on how the initial contract is written. If the fund leverage target is "soft," meaning that it is written as $D_0(B) \leq \overline{D}_0$, and if the carry interest share is set according to equation (10) under the presumption of fund leverage being set at the constrained maximum, then the GP may have *ex post* incentives to shade fund leverage to the low side. Doing so provides the GP the opportunity to increase fees at the expense of net-of-fee fund returns.

If the LP recognizes this incentive and is concerned that the GP will act on it, the LP can modify contract language to contain a "hard" fund leverage target that cannot deviate from $D_0(B) = \overline{D}_0$ unless approved by the LP. Interestingly, both types of contracts are known to be written in the PE industry.

The base-case of α=.02 exceeds α^{Min}=.0088, ensuring that ρ>0 and that fund leverage is set *ex ante* at the target. Going forward I round that target to $\overline{D}_0 = 65.00$. I now vary the carried interest hurdle value, ψ, within a reasonable range to analyze what the model has to say about actual incentive contracting practices. Using base-case parameter values and varying ψ between .05 and .12, in the left-hand panel of Table 5 I report the endogenously determined carried interest share, $\bar{\rho}$. This share value is seen to range between .20 and .26, which is only slightly above the 20.0% share observed in practice. At ψ=.09 (the median value in the PERE data), I obtain $\bar{\rho} = .2280$. It is worth noting that α=.0183 results in $\bar{\rho} = .200$ when ψ=.09. These outcomes lend support not only to observed PE incentive fee contracting practices, but are also consistent with return targeting practices that have been documented in the sector.

In the right-hand panel of Table 5 I consider incentives to deviate from the *ex ante* contract when target leverage is stipulated as a soft constraint. When ψ=.11 or .12, the GP has no incentive to deviate to the low side. However, at lower ψ values, the GP has an incentive to deviate to the lower incentive compatible leverage levels. Deviating *ex post* increases GP incentive fees to be above those obtained given a hard fund leverage

[24] I have previously verified that this outcome does not satisfy necessary constrained optimality conditions *ex ante*.

target, and decreases expected returns to the LP to be below the target. The fact that fees only increase by a small amount when the GP behaves opportunistically, which could come at some future cost to the GP, suggests that incentives to deviate may be low, and that as a consequence fund leverage will generally cluster at $\bar{D}_0 = 65.00$. This is precisely what I see in the data, offering additional support to the analysis.

Table 5

Varying the Baseline Incentive Contract Parameters
Given LP Return and Fund Leverage Targeting

Base Case: α=.02

	Panel A: *Ex Ante* Contracting				Panel B: *Ex Post* Leverage Shading			
ψ	$\bar{\rho}$	$\bar{D}_0$	Φ^V	λ^N	$\bar{\rho}$	$Min\{\bar{D}_0, D_0^*\}$	Φ^V	λ^N
.05	.2003	65.00	16.56	.1672	.2003	53.13	16.95	.1490
.06	.2062	65.00	16.56	.1672	.2062	56.29	16.81	.1539
.07	.2127	65.00	16.56	.1672	.2127	58.94	16.70	.1580
.08	.2200	65.00	16.56	.1672	.2200	61.17	16.63	.1614
.09	.2280	65.00	16.56	.1672	.2280	63.09	16.57	.1644
.10	.2370	65.00	16.56	.1672	.2370	64.79	16.56	.1669
.11	.2471	65.00	16.56	.1672	.2471	65.00	16.56	.1672
.12	.2583	65.00	16.56	.1672	.2583	65.00	16.56	.1672

Notes: The baseline incentive contract is considered given the constrained optimization of GP incentive fees. Fund leverage and carried interest share are determined endogenously. Alternative carried interest hurdles are considered, ranging from ψ=.05 to ψ=.12. In addition to carried interest share, $\bar{\rho}$, and fund leverage, $\bar{D}_0$, GP incentive fees, Φ^V, and LP net-of-fee return, λ^N, are determined. Panel A shows constrained optimization results, while Panel B shows how the GP may have incentives to hold up the LP by shading leverage to the low side after the contract is signed. Shaded values are those outcomes with leverage less than the target of 65.00. Base case parameter values are: V_0=100; r=.02 : μ=.10; α=.02; σ=.175; k=.30; T=6.

Given the base-case results reported in RHS of Table 5, note the "hinge" in ψ at around .10 (indicated by the shaded v. unshaded regions). This, along with $\bar{\rho}$ close to .20, provides insight into the emergence of the 9-20 contract. Contract parameter values less than or equal to ψ=.10 provide increasingly strong incentives for the GP to deviate from the *ex ante* contract. In other words, the model indicates that GP and LP may have mutually settled in on a 9.0% carried interest hurdle rate because lower hurdle rates increase GP incentives to behave opportunistically, while higher rates benefit the LP at the GP's expense.

In Table 6 the baseline incentive contract is examined after introducing GP skill heterogeneity. In panel A I consider the lower-skill case of α=.015, and in panel B I consider the higher-skill case of α=.030. As with the base-case α=.020 both the target return and fund leverage constraints bind, with the LP breaking even. When

GP skill level is lower (panel A), the interest carry share required to hit the target return is well below the 20.0% mark commonly used in practice, while in the higher-skill case (panel B) it is significantly above the 20.0% share mark. The implication is that the lower-skill GP is overcompensated with the 9-20 contract, which in turn implies an inability of the GP to meet the LP's target return. According to the model, with return targeting these lower-skill GP's will have to exit the market if the standard baseline contract is implemented. On the other hand, as seen in panel B higher-skill GPs are undercompensated given the standard 9-20 contract. They may also exit, or threaten to exit the market, unless a supplemental compensation arrangement can be implemented.

Table 6

Varying the Baseline Incentive Contract Parameters
Given LP Return and Fund Leverage Targeting

	Panel A: α=.015				Panel B: α=.03			
ψ	$\bar{\rho}$	$\bar{D}_0$	Φ^V	λ^N	$\bar{\rho}$	$\bar{D}_0$	Φ^V	λ^N
.05	.1217	65.00	9.27	.1672	.3237	65.00	31.08	.1672
.06	.1254	65.00	9.27	.1672	.3324	65.00	31.08	.1672
.07	.1296	65.00	9.27	.1672	.3419	65.00	31.08	.1672
.08	.1342	65.00	9.27	.1672	.3525	65.00	31.08	.1672
.09	.1394	65.00	9.27	.1672	.3642	65.00	31.08	.1672
.10	.1452	65.00	9.27	.1672	.3772	65.00	31.08	.1672
.11	.1516	65.00	9.27	.1672	.3918	65.00	31.08	.1672
.12	.1589	65.00	9.27	.1672	.4080	65.00	31.08	.1672

Notes: The baseline incentive contract is considered given the constrained optimization of GP incentive fees and carried interest share. Two different GP skill levels are considered: α=.015 and α=.030. Alternative carried interest hurdles are also considered, ranging from ψ=.05 to ψ=.12. In addition to carried interest share, $\bar{\rho}$, and fund leverage, $\bar{D}_0$, GP incentive fees, Φ^V, and LP net-of-fee return, λ^N, are determined. Base case parameter values are: V_0=100; r=.02 : μ=.10; σ=.175; k=.30; T=6.

Thus, the model predicts that only GPs in a fairly tight range around α=.02 will, in a world in which LPs target returns, remain eligible as well as willing to adopt the standard 9-20 baseline contract. Higher-skill GPs in particular will be motivated to explore an alternative or augmented contract that can accommodate GP skill heterogeneity. As documented earlier, such a contract is known to exist in the form of carried interest with a catch-up fee provision. I turn to this topic in the next sub-section.

VI.B. GP Skill Heterogeneity and Catch-Up Fees

Modeling thus far provides insight into the emergence of the baseline 9-20 carried interest contract. But the robustness of the simple 9-20 contract over time and across funds is questionable. GP skill heterogeneity is a particularly important margin along which the invariant 9-20 contract seems too rigid for my baseline model to fit the data.

Recalling prior empirical results showing significant variation in the PERE catch-up rate, I will now augment the baseline contract to incorporate the catch-up fee provision. Catch-up fees result when, after the full payment of preferred returns to LP's, disproportionately high profits accrue to the GP until the GP's total profit share "catches up" to the carried interest share percentage as applied to total equity profits.

The catch-up fee provision formally works as follows. Denote the catch-up rate as ξ, $\rho<\xi\leq 1.0$. The catch-up rate applies, and only applies, once liquidated fund proceeds of V_T exceed prioritized debt payouts of B to the lender and preferred interest payouts of $e^{\psi T}E_0$ to the LP. The sum of those two quantities was previously denoted by χ_0. In the catch-up region, where the catch-up rate applies, a proportion ξ on every excess liquidation dollar is paid to the GP until the GP has earned 100*ρ percent of all profits in excess of B. If and when catch-up is complete, the GP goes back to earning 100*ρ percent of all remaining profits. Complete catch-up, when achieved, reduces the LP's preferred interest payout from $e^{\psi T}E_0$ to $(1\text{-}\rho)e^{\psi T}E_0$.

Denote the upper bound of the catch-up region by X_0. This upper bound is determined such that,

$$[\mathrm{X}_0 - \chi_0]\xi = [\mathrm{X}_0 - B]\rho \tag{12a}$$

This is easily solved for X_0, and written as,

$$\mathrm{X}_0 = B + \frac{\xi E_0 e^{\psi T}}{\xi - \rho} \tag{12b}$$

Note that, in contrast to χ_0, which serves as the lower bound of the catch-up region and only contains one incentive fee contracting variable, ψ, X_0, the upper bound of the region, is a function of all three contracting variables, ψ, ρ, and ξ.

With the catch-up region defined, incentive fees expected to be paid out to the GP at time T can be determined. This expectation is,

$$\Phi_{CU}^{V}(B,\xi;\psi,\rho) = \xi \int_{\chi_0}^{\infty}(\tilde{V}_T - \chi_0) f(\tilde{V}_T|V_0) d\tilde{V}_T - (\xi - \rho) \int_{\mathrm{X}_0}^{\infty}(\tilde{V}_T - \mathrm{X}_0) f(\tilde{V}_T|V_0) d\tilde{V}_T \qquad \text{(13a)}$$

Solving the integrals in (13a) results in the following relation:

$$\Phi_{CU}^{V}(B,\xi;\psi,\rho) = \xi \left[V_0 e^{(\mu+\alpha)T} N[h_1] - \chi_0 N[h_2]\right]$$

$$-(\xi - \rho)\left[V_0 e^{(\mu+\alpha)T} N[m_1] - \mathrm{X}_0 N[m_2]\right] \qquad \text{(13b)}$$

where h_1 and h_2 are as previously defined in (2b') and

$$m_1 = \frac{ln\left[{}^{V_0}/_{\mathrm{X}_0}\right] + \left((\mu+\alpha) + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \; m_2 = m_1 - \sigma\sqrt{T} \qquad \text{(13b')}$$

Note that the addition of a catch-up fee provision negates separation that previously applied. Importantly, however, linear homogeneity of Φ_{CU}^{V} in V_0 and B is retained. I also note that $h_1 > m_1$ and $h_2 > m_2$ due to $\chi_0 < \mathrm{X}_0$. This in turn implies $N[h_1] > N[m_1]$ and $N[h_2] > N[m_2]$.

Unconstrained incentive compatible fund leverage can now be calculated by optimizing the incentive fee equation in (9b) with respect to B. Doing so results in,

$$-\xi N[h_2]\frac{\partial \chi_0}{\partial B} + (\xi - \rho)\frac{\partial \mathrm{X}_0}{\partial B} = 0 \qquad \text{(14a)}$$

which can be rewritten as,

$$1 - e^{(\psi - r)T}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right] = \frac{\rho N[m_2]}{\xi[N[m_2] - N[h_2]]} < 0 \qquad \text{(14b)}$$

Note that the LHS of (14b) corresponds exactly to the optimality condition that applies in the baseline contracting case (see Proposition 2). Now, after adding the catch-up provision into the incentive fee contract, the RHS of (14b) is negative rather than zero as in the baseline case. With this, the following proposition characterizes the GP's unconstrained incentive compatible fund capital structure in the presence of a catch-up provision:

Proposition 4 (Unconstrained Fund Leverage Choice with Catch-Up): An incentive compatible fund capital structure exists with a catch-up provision characterized by ξ, $\rho < \xi \leq 1$. There are three cases to

consider: 1) When $\boldsymbol{\psi > r}$, and when $\boldsymbol{1 - e^{(\psi - r)T} < \frac{\rho N[m_2]}{\xi[N[m_2] - N[h_2]]}}$ as evaluated at *B*=0, incentive compatible fund leverage is such that $\boldsymbol{B^*_{CU} > 0}$. In this case $\boldsymbol{B^*_{CU} < B^*}$ for all *ψ*>*r*. 2) When $\boldsymbol{\psi > r}$ and $\boldsymbol{1 - e^{(\psi - r)T} \geq \frac{\rho N[m_2]}{\xi[N[m_2] - N[h_2]]}}$ at *B*=0, $\boldsymbol{B^*_{CU} = 0}$. 3) When $\boldsymbol{\psi \leq r}$, $\boldsymbol{B^*_{CU} = B^* = 0}$.

<u>Proof</u>: See Appendix A

Because the LHS of equation (14b) is strictly increasing in B, it follows that $B^*_{CU} \leq B^*$. Thus, all else equal, unconstrained incentive compatible fund debt level with a catch-up provision is typically lower, not higher, than fund leverage realized under the standard incentive fee contract.

This result is not necessarily intuitive, and can be better seen by rearranging equation (14b) as follows below, and then comparing it to equation (3):

$$\frac{e^{\psi T}}{1 - \frac{\rho N[m_2]}{\xi[N[m_2] - N[h_2]]}} = \frac{e^{rT}}{\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]} = \frac{1}{\frac{\partial D_0}{\partial B}} \tag{14c}$$

The two terms on the RHS of (14c) exactly matches the RHS of equation (3), showing the marginal increase in the cost of debt as fund leverage increases. In the baseline contract, this increase in the cost of fund leverage is compared to the benefit associated with decreasing the equity base from which the preferred return is calculated (see the LHS of equation (3)). With a catch-up provision the LHS of (3) is modified to include the denominator seen in the LHS of (14c). This denominator is always greater than one since $\frac{\rho N[m_2]}{\xi[N[m_2] - N[h_2]]} < 0$. Thus the LHS of (14c) is always less than the LHS of equation (3), implying the benefits of fund leverage with a catch-up fee provision are less than the benefits associated with the baseline contract. This follows because the catch-up provision allows the GP to recapture preferred dividends allocated on a priority basis, which lowers the marginal cost of LP preferred equity. Given realistic parameter values, it is further the case that the marginal cost of preferred interest erodes more quickly, and with greater certainty, as the catch-up rate, ξ, increases.

This is a very interesting relation that to my knowledge has not been explored in the literature. In contrast to the baseline contract, where fund leverage is increasing GP skill, fund leverage decreases in GP skill when the baseline contract is augmented to include a catch-up provision. This can be seen in equation (14b), where the LHS of the equality is independent of α, while the RHS is decreasing in α. Intuitively, all else equal, and for a given catch-up rate, ξ, higher-skill GPs stand a greater chance of recovering the preferred equity payout than lower-skill GPs, which reduces the cost of equity capital to reduce leverage. Compounding this relation, however, is that higher-skill GPs are also in a better position to implement higher catch-up rates, with fund leverage increasing in ξ (again see equation (14b)).

The following corollary now states two useful relations that will aid in subsequent analysis.

<u>Corollary to Proposition 4</u>: $\frac{\partial \Phi_{CU}^{V}}{\partial \xi} > 0$ and $\frac{\partial \lambda}{\partial \xi} < 0$. In particular, $\frac{\partial \Phi_{CU}^{V}}{\partial \xi} = V_0 e^{(\mu+\alpha)T}\left[N[h_1] - N[m_1]\right] - \chi_0\left[N[h_2] - N[m_2]\right]$ and $\frac{\partial \lambda}{\partial \xi} = \frac{\frac{-\partial \Phi_{CU}^{V}}{\partial \xi}}{T(\mathcal{E}_T - \Phi_{CU}^{V})}$.

<u>Proof</u>: See Appendix A.

Like the effect of the carried interest share, ρ, GP incentive fees are increasing in the catch-up rate, ξ. And like the carried interest share, the catch-up rate is capped at one. But the effects at the upper bound differ substantially. The carried interest share is a rather blunt incentive compensation instrument, where higher shares extract profits in a clear, linearly homogeneous manner, and in the limit (ρ=1) take all profits in excess of the preferred interest payout. The effects of the catch-up rate are, in contrast, more subtle and less substantial. A catch-up rate of one is only in effect within a given range of profits, with profit sharing reverting back to the carried interest share after "excess" preferred interest payouts are recovered by the GP. Thus, the effect of $\xi = 1.0$ (which is often quoted in the general PE literature) is not as extreme as it is for ρ=1.0 (which is never observed in practice). As for the complementary comparative static, $\frac{\partial \lambda}{\partial \xi}$, what is good for GP fees is bad for LP net-of-fee return. Hence $\frac{\partial \lambda}{\partial \xi} < 0$.

Analysis in this sub-section has not yet imposed the constraint that the GP meet the LP's return and fund leverage target. To do so I specify the following constrained optimization problem:

$$\underset{B,\xi}{Max}\ \Phi_{CU}^{V}(B,\xi;\psi,\rho) = \xi\left[V_0 e^{(\mu+\alpha)T} N[h_1] - \chi_0 N[h_2]\right]$$

$$-(\xi - \rho)\left[V_0 e^{(\mu+\alpha)T} N[m_1] - \mathrm{X}_0 N[m_2]\right] \qquad (15)$$

$$\text{s.t. } 0 \leq D_0(B) \leq \overline{D}_0,\ \lambda_{CU}^N(B,\rho) \geq \lambda^*,\ \rho < \xi \leq 1.0 \tag{15a}$$

In this case the catch-up rate in addition to fund leverage is endogenously determined, with baseline contract parameters ψ and ρ taken as given (as is generally the case in practice).

There are four cases to consider in solving the problem, all of which depend critically on GP skill level, α. The first case requires identifying a bound for alpha, $\dot{\alpha}$, such that low-skill GPs endowed with $\alpha \leq \dot{\alpha}$ are unable to implement the catch-up fee provision. The second case considers moderately skilled GPs that are capable of implementing the catch-up fee contract, but not skilled enough to implement a full catch-up fee contract with $\xi = 1.0$. This requires finding an additional bound, $\ddot{\alpha} > \dot{\alpha}$, such that at $\alpha = \ddot{\alpha}$ full catch-up is just feasible. In this range an intermediate catch-up rate of $\rho < \xi < 1.0$ obtains that varies directly with alpha. The third case is that of a highly skilled GP endowed with $\alpha, \ddot{\alpha} \leq \alpha < \dddot{\alpha}$. In this range the GP is able to implement the full catch-up fee contract while also meeting the target return constraint. Fund leverage is less than the targeted upper bound, however. Finally, in the fourth case the GP is endowed with extreme skill, with $\alpha \geq \dddot{\alpha}$, such that the debt lower bound of B=0 is binding. It turns out that alpha levels in this range far exceed those that fit the calibrated data.[25] As a consequence, although this fourth case is feasible in theory, I will not pursue it any further.

To solve (15) I form the Lagrangian, with KKT conditions as follows:

$$\mathcal{L}(B,\xi) = -\Phi^V(\rho, B; \psi) + \mu_1\big(\lambda^* - \lambda^N(B,\rho)\big) + \mu_2\big(D_0(B) - \overline{D}_0\big) + \mu_3(\xi - 1) \tag{16}$$

and where $\mu_1, \mu_2, \mu_3 \geq 0$ denote Lagrange multipliers.

Prior to stating solutions to the constrained optimization problem it is also useful to recall the following relation:

$$\Phi_{CU}^V = \mathcal{E}_T - e^{\lambda^* T}[E_0 + \Phi^F] \tag{7a}$$

[25] Using base case parameter values I find that $\dddot{\alpha} = .111$ for B to hit a lower bound of zero, which is a value that far exceeds GP skill levels implied in the PERE data.

This is simply a rearrangement of equation (7), along with constraining λ^N to equal λ^*. This relation will be used to locate solutions when analyzing KKT conditions according to equation (16).

With this, I am now in a position to state the constrained efficient solution to the catch-up fee contracting problem.

<u>Proposition 5 (The Constrained Catch-Up Fee Contracting Problem)</u>: For empirically-based parameter value ranges established previously, there are three solution ranges to consider. To bracket the solution ranges, let $\dot{\alpha}$ be that $\alpha > 0$ with $\lambda = \lambda^*$ and $D_0 = \overline{D}_0$ given $\xi = \rho$, and let $\ddot{\alpha}$ be that $\alpha > \dot{\alpha}$ at which $\lambda = \lambda^*$ and $D_0 = \overline{D}_0$ given $\xi = 1.0$. The contracting solution ranges are as follows: 1) When $\alpha < \dot{\alpha}$, no catch-up provision is included the incentive contract; 2) When $\dot{\alpha} \leq \alpha < \ddot{\alpha}$, both the target return and fund leverage constraints bind, with ξ chosen to satisfy equation (7a); 3) When $\alpha \geq \ddot{\alpha}$, full catch-up $\xi = 1.0$ is implemented, along with $\lambda = \lambda^*$. $D_0(B) < \overline{D}_0$ is chosen to satisfy equation (7a).

<u>Proof</u>: See Appendix A.

For $\alpha \leq \dot{\alpha}$ there is no $\xi > \rho$ that can simultaneously satisfy the target return and fund leverage constraints for the given ψ and ρ, while also resulting in GP incentive fees that are positive in expectation. Implicit in this result is that, for empirically-based parameter values established previously, and because $\alpha > 0$, LP returns are increasing in fund leverage at the fund leverage constraint. This causes the low-skill LP to lever the fund up to the $D_0 = \overline{D}_0$ constraint in a (failed) attempt to meet the LP's return target.

In all cases for which the catch-up fee provision is implementable, constrained optimal GP incentive fees are determined as a two-part tariff. For $\dot{\alpha} \leq \alpha < \ddot{\alpha}$, the GP optimizes incentive fees by setting $D_0 = \overline{D}_0$, and then using equation (7a) to find the catch-up rate, ξ, to meet the LP's return target. The GP optimizes over the catch-up rate rather than fund leverage, since the LP's return target is more sensitive to reductions in fund leverage than it is to increases in the catch-up rate. In this case the Lagrange multipliers are of the exact same form as in the baseline contracting case, except now incentive fees paid to the GP incorporate the fee-increasing effects of the catch-up provision. This augmented contract, which reduces fund leveraging incentives of the GP, has the effect of reducing shadow costs at both the return target and fund leverage constraint. For $\alpha \geq \ddot{\alpha}$, the catch-up rate binds at $\xi = 1.0$. Here the GP finds $D_0(B) < \overline{D}_0$ that satisfies (7a). Because fund leverage moves inversely with α, the GP reduces leverage to increase fees. This decreases LP returns, with the GP reducing fund leverage to the point at which $\lambda^N = \lambda^*$.

For the empirically calibrated parameter values used herein, the target return constraint will always be binding. Meeting this return constraint causes the GP to implement fund leverage that is higher than preferred in the absence of the constraint. Simultaneously satisfying GP incentive compatibility and LP benchmark return requirements was feasible in the baseline contract case due to separation between ψ and ρ. Separation is lost when the catch-up provision is added, however.

Note that *ex post* leverage shading incentives exist when $\alpha \geq \ddot{\alpha}$. This is due to the inverse relation between GP incentive compatible fund leverage choice and skill when implementing the catch-up contract. When the LP recognizes this effect, it may seek a hard fund leverage target to establish credible commitment on the part of the GP.

Consider now calibrated modeling results using base-case parameter values, along with the previously identified target return of λ^*=.1672 and target fund leverage of $\overline{D}_0$=65.00. With these values, from equation (7a) it immediately follows that $\dot{\alpha} = .0183$ and $\ddot{\alpha} = .0258$. As a result I will examine GP skill levels with alpha equal to or exceeding 2.0%, as the catch-up fee contract is not implementable for $\alpha \leq .0183$. Table 7 reports the constrained efficient fund capital structure and incentive fee results for GP's endowed with varying skill levels, along with several other contracting scenarios.

GP alpha values of .020 to .040 are seen across the top of the table. Four contracting cases are examined, as identified in panels A through D. In all cases, ψ=.09 and ρ=.20. For each case, four quantities are calculated: 1) The catch-up rate, ξ; 2) Fund leverage, $D_0(B)$; 3) GP incentive fees, Φ^V_{CU}; and 4) LP return, λ^N_{CU}. To establish a benchmark, panel A considers the baseline contract without a catch-up fee provision and with a soft leverage constraint, $D_0(B) \leq \overline{D}_0$. Panel B presents the constrained efficient catch-up fee contracting outcome based on the application of proposition 5. This case corresponds to there being a soft fund leverage constraint, $D_0(B) \leq \overline{D}_0$. Panel C considers the case in which the LP negotiates for a hard leverage constraint, which, given parameter values applied herein, discourages GP opportunism and in certain cases increases LP returns relative to the soft constraint case. Lastly, panel D recognizes additional LP bargaining power. Here the LP retains the hard fund leverage constraint, and further negotiates a lower catch-up rate than the rate the GP prefers through

Table 7

Catch-Up with LP Return and Fund Leverage Targeting

Line Item	α=.02	α=.025	α=.03	α=.04
Panel A				
Baseline Contract: ψ.09, ρ=.20				
(Soft Leverage Target)				
ξ	--	--	--	--
$D_0 = \bar{D}_0$	63.09	65.00	65.00	65.00
Φ^V	14.54	15.78	17.07	19.76
λ^N	.1674	.1790	.1873	.2029
Panel B				
Catch-Up Contract:				
Constrained Efficient Implementation by GP				
ξ	.228	.480	1.0	1.0
D_0	65.00	65.00	61.57	54.48
Φ_{CU}^V	16.54	23.79	27.00	31.40
λ_{CU}^N	.1672	.1672	.1672	.1672
Panel C				
Catch-Up Contract:				
Hard Leverage Target				
ξ	.228	.480	1.0	1.0
$D_0 = \bar{D}_0$	65.00	65.00	65.00	65.00
Φ_{CU}^V	16.54	23.79	26.28	29.53
λ_{CU}^N	.1672	.1672	.1744	.1904
Panel D				
Catch-Up Contract:				
LP with Additional Bargaining Power				
ξ	--	.300	.500	.750
D_0	65.00	65.00	65.00	65.00
Φ_{CU}^V	14.52	21.62	25.52	29.40
λ_{CU}^N	.1703	.1705	.1755	.1906

Notes: Various quantities are displayed for alphas that range between .020 and .040. Each panel contains four quantities: i) Catch-up rate, ξ; ii) Fund leverage, D_0; iii) GP incentive fees, Φ_{CU}^V; and iv) LP net-of-fee return, λ_{CU}^N. Panels differ by details of the fund offering documents. Panel A considers the baseline incentive contract; Panel B considers the augmented contract with a catch-up fee provision along with a soft leverage constraint; Panel C considers the augmented contract with a catch-up fee provision along with a hard leverage constraint; Panel D considers the augmented contract with a catch-up fee provision, a hard leverage constraint, and a catch-up rate set by the LP. Base case parameter values are: V_0=100; r=.02 : μ=.10; σ=.175; k=.30; T=6; ψ=.09, ρ=.20.

its constrained optimization.

In the baseline contracting case considered in panel A, GP incentive compatible fund leverage increases in α. Only when α=.02 does fund leverage decrease below the leverage target of 65.00. GP fees increase in α even though leverage is capped, due to increasing GP skill causing greater carried interest. Lastly, LP returns increase in GP skill. With ψ and ρ fixed, given the baseline contract the GP has no other instruments with which to extract fees to decrease LP return.

Panel B implements the constrained efficient contract as described in Proposition 5. Alpha equal to .02 or .025 falls in the moderate-skill range, which according to Proposition 5 results in fund leverage at the cap of $\overline{D}_0 = 65.00$. For both moderate alpha values, Lagrange multipliers μ_1 and μ_2 are confirmed to be positive, while μ_3=0. Alpha equal to .03 or .04 falls in the high-skill range, which according to Proposition 5 results in fund leverage below the cap of $\overline{D}_0 = 65.00$. Fund leverage of 61.57 and 54.48, respectively, are well within normal ranges seen in the PERE data (see Tables 2 and 3). Although the fund leverage constraint no longer binds in this high-skill range, the catch-up rate does bind at $\xi = 1.0$. Finally, note the substantial increase in GP incentive fees at all skill levels when moving from the baseline contract to the augmented catch-up fee contract. The increase in fees are particularly large – at about 60 percent – for the high-skill GPs that are in a position to impose the full catch-up rate and decrease fund leverage.

Panel C is populated based on the assumption of a hard leverage contracting provision. The imposition of this provision affects only the high-skill GPs that prefer fund leverage below the target (as seen in Panel B). In these cases the increase in fund leverage benefits the LP, whose returns increase from the target of .1672, at the expense of GP incentive fees. The decrease in GP fees is not dramatic, however, particularly in the case of α=.03.

Panel D is constructed assuming that the LP has some bargaining power that reduces the catch-up rate below the rate preferred by the GP. The catch-up rate reductions are fairly modest relative to the GP-preferred rates. A presumption of a hard fund leverage provision is also maintained. As seen in the table, these catch-up rate reductions do reduce incentive fees of moderately-skilled GPs by approximately 10 percent, while there is little impact on fees for the higher skill GPs. Analogous offsetting impacts are also noted in LP returns.

In summary, I explain the catch-up fee provision as a mechanism utilized by higher-skill GPs to hit LP return targets without having to adjust the standard carried interest contract fee variables. Lower-skill GPs either utilize the standard baseline contract or exit the market. Catch-up fees vary directly with skill level, with GPs trading off higher fund leverage, within the constrained range, in order to maximize the catch-up fee rate. This approach works in constrained equilibrium because, generally speaking, LP fund performance responds more to leverage than to changes in the catch-up rate. High-skill GPs negotiate for the maximum catch-up rate of $\xi = 1.0$, and then decrease fund leverage because of the low cost of equity capital.

Empirical predictions include: 1) Low-skill GPs are unable to negotiate catch-up fee contracts. This is seen in the PERE data, where a significant number of funds do not include catch-up fee provisions. For lower risk, lower alpha PERE funds such as core and core-plus, catch-up fee provisions are rarely included; 2) For moderately skilled GPs, fund leverage clusters at the target upper bound, with more variation in the catch-up rate that moves directly with GP skill. Leverage clustering along with significant variation in catch-up rates as described are clearly seen in the PERE data; 3) Higher-skill GPs cluster at the full catch-up rate, with an inverse relation between skill level and fund leverage. Consistent with the existence of limited alpha, few PERE funds implement the full catch-up rate and fund leverage levels are high at an average of over 63% and with an interquartile range of 60% to 70%.. This in contrast to PE buyout funds, where alpha estimates are higher, full catch-up fee rates are common, and fund leverage levels are lower on average, at 49%, with a relatively wide interquartile range of 37% to 62% (see, e.g., Brown, Harris and Munday (2021)).

VII. Conclusion

A model of closed-end PE fund capital structure is developed in this paper to explain the carried interest compensation contract. In the model, GP's implement fund leverage to optimize incentive fees while also satisfying target return objectives of LP's. To motivate the modeling, fee, leverage and target return data from private equity real estate funds are analyzed. Steps in the modeling process include developing a model of debt funding cost that pits alpha against costs of financial distress. Endogenous upper bounds on fund leverage are a byproduct of the model, where I show that GPs with convex carried interest payoff functions limit fund leverage even when financial distress costs are zero. Parameter values are selected after a thorough analysis and careful matching to the relevant data. Given standard incentive fee contracting terms, modeled capital structures closely match those observed empirically. To address the issue of GP skill heterogeneity the model is extended to consider catch-up fee provisions. I first endogenize LP return targeting preferences, and then show how such provisions arise endogenously as a mechanism that allows higher-skill GPs to extract higher fees while also satisfying LP return target constraints.

References

Andonov, Aleksandar, Rob M.M. Bauer and K.J. Martijn Cremers, 2017, Pension Fund Asset Allocation and Liability Discount Rates, *Review of Financial Studies* 30, 2555-2595.

Andonov, Aleksandar and Joshua D. Rauh, 2019, The Return Expectations of Institutional Investors, Unpublished Manuscript, Stanford University.

Andrade, Gregor and Steven N. Kaplan, 1998, How Costly is Financial (Not Economic) Distress? Evidence from Highly Leveraged Transactions that Became Distressed, *Journal of Finance* 53, 1443-1493.

Arnold, Thomas R., David C. Ling and Andy Naranjo, 2017, Waiting to be Called: The Impact of Manager Discretion and Dry Powder on Private Equity Real Estate Returns, *Journal of Portfolio Management* 43 (Fall), 23-43.

Axelson, Ulf, Morten Sorensen and Per Stromberg, 2014, Alpha and Beta of Buyout Deals: A Jump CAPM for Long-Term Illiquid Investments, Unpublished Manuscript, Columbia University.

Axelson, Ulf, Per Stromberg and Michael S. Weisbach, 2009, Why Are Buyouts Levered? The Financial Structure of Private Equity Funds, *Journal of Finance* 64, 1549-1582.

Ben-David, Itzhak, Justin Birru and Andrea Rossi, 2020, The Performance of Hedge Fund Performance Fees, *NBER WP #27454.*

Berk, Jonathon B. and Richard C. Green, 2004, Mutual Fund Flows and Performance in Rational Markets, *Journal of Political Economy* 112, 1269-1295.

Bodnaruk, Andriy and Andrei Siminov, 2016, Loss-Averse Preferences, Performance, and Career Success of Institutional Investors, *Review of Financial Studies* 29, 3140-3176.

Bollinger, M. and J. Pagliari Jr., 2019, Another Look at Private Real Estate Returns by Strategy, *Journal of Portfolio Management* 45, 1-18.

Brown, Greg, et al., 2020, Debt and Leverage in Private Equity: A Survey of Existing Results and New Findings, *Working Paper*, Institute for Private Capital, University of North Carolina – Chapel Hill.

Brown, Greg, Robert Harris and Shawn Munday, 2021, Capital Structure and Leverage in Private Equity Buyouts, *Journal of Applied Corporate Finance* 33, 42-58.

Brown, Jeffrey R. and David W. Wilcox, 2009, Discounting State and Local Liabilities, *American Economic Review* 99, 538-542.

Carpenter, Jennifer, 2000, Does Option Compensation Increase Managerial Risk Appetite?, *Journal of Finance* 55, 2311-2331.

Chung, Ji-Woong, Berk A. Sensoy, Lea Stern and Michael S. Weisbach, 2012, Pay for Performance from Future Fund Flows: The Case of Private Equity, *Review of Financial Studies* 25, 3259-3304.

Cochrane, John H., 2005, The Risk and Return of Venture Capital, *Journal of Financial Economics* 75, 3-52.

Demiroglu, Cem and Christopher M. James, 2010, The Role of Private Equity Group Reputation in LBO Financing, *Journal of Financial Economics* 96, 306-330.

Franzoni, F. and J. Marin, 2006, Pension Plan Funding and Stock Market Efficiency', *Journal of Finance* 61, 921-956.

Goetzmann, William N., Elise Gourier and Ludovic Phalippou, 2019, How Alternative Are Private Markets, *Working Paper*, Yale University.

Gompers, Paul, Steven N. Kaplan and Vladimir Mukharlyamov, 2016, What Do Private Equity Firms Say They Do? *Journal of Financial Economics* 121, 449-476.

Grinblatt, Mark, and Sheridan Titman, 1989, Adverse Risk Incentives and the Design of Performance-Based Contracts, *Management Science* 35, 807-822.

Gupta, Arpit and Stijn Van Nieuwerburgh, 2021, Valuing Private Equity Investments Strip by Strip, forthcoming *Journal of Finance.*

Jenkinson, Tim and Rudiger Stucke, 2011, Who Benefits from Leverage in LBOs?, *Working Paper*, Oxford University.

Jensen, Michael C., 1989, Eclipse of the Public Corporation, *Harvard Business Review* (Sept-Oct), 1-31.

Kaplan, Steven N. and Josh Lerner, 2010, If It Ain't Broke: The Past, Present and Future of Venture Capital, *Journal of Applied Corporate Finance* 22, 3 6-47.

Korteweg, Arthur, 2019, Risk Adjustment in Private Equity Returns, *Annual Review of Financial Economics* 11, 131-152.

Lan, Yingcong, Neng Wang and Jinqiang Yang, 2013, The Economics of Hedge Funds, *Journal of Financial Economics* 110, 300-323.

Lerner, Josh, Antoinette Schoar and Wan Wongsunwai, 2007, Smart Institutions, Foolish Choices: The Limited Partner Performance Puzzle, *Journal of Finance* 62, 731-764.

Lim, Jongha, Berk A. Sensoy and Michael S. Weisbach, 2016, Indirect Incentives of Hedge Fund Managers, *Journal of Finance* 71, 871-918.

Lopes-de-Silanes, Florencio, Ludovic Phalippou and Oliver Gottschalg, 2015, Giants at the Gate: Investment Returns and Diseconomies of Scale in Private Equity, *Journal of Financial and Quantitative Analysis* 50, 377-41.

Merton, Robert C., 1974, On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *Journal of Finance* 29, 449-470.

Metrick, Andrew and Ayako Yasuda, 2010, The Economics of Private Equity Funds, *Review of Financial Studies* 23, 2303-2341.

Novy-Marx, Robert and Joshua Rauh, 2009, The Liabilities and Risks of State-Sponsored Pension Plans, *Journal of Economic Perspectives* 23, 191-210.

Pagliari, Joseph L., 2020, Real Estate Returns by Strategy: Have Value-Add and Opportunity Funds Pulled Their Weight?, *Real Estate Economics* 48, 89-134.

Panageas, Stavros and Mark M. Westerfield, 2009, High-Water Marks: High Risk Appetites? Convex Compensation, Long Horizons, and Portfolio Choice, *Journal of Finance* 64, 1-36.

Phalippou, Ludovic and Oliver Gottschalg, 2008, The Performance of Private Equity Funds, *Review of Financial Studies* 22, 1747-1776.

Phalippou Ludovic, 2018, *Private Equity Laid Bare*, www.pelaidbare.com.

Phalippou, Ludovic, 2021, An Inconvenient Fact: Private Equity Returns and the Billionaire Factory, forthcoming, *Journal of Investing.*

Riddiough, Timothy J., 2021, Pension Funds and Private Equity Real Estate: History, Performance, Pathologies, Risks, forthcoming, *Real Estate and Macroeconomics,* Elgar.

Robinson, David T. and Berk A. Sensoy, 2013, Do Private Equity Fund Managers Earn Their Fees? Compensation, Ownership, and Cash Flow Performance, *Review of Financial Studies* 26, 2760-2797.

Ross, Stephen A., 2004, Compensation, Incentives, and the Duality of Risk Aversion and Riskiness, *Journal of Finance* 59, 207-225.

Shilling, James D. and Charles H. Wurtzebach, 2012, Is Value-Added and Opportunistic Real Estate Investing Beneficial? If So, Why? *Journal of Real Estate Research* 34, 429-461.

Sorensen, Morten, Neng Wang and Jinqiang Yang, 2014, Valuing Private Equity, *Review of Financial Studies* 27, 1977-2021.

Appendix A

Proofs to Propositions and Corollaries

For all of the proofs subscripts are suppressed wherever doing so does not introduce any ambiguity into the meaning of the variables.

Proof of Proposition 1: Starting with equation (1b), after doing some algebra and using the known formula for the standard normal pdf, I have that $\frac{\partial D_0}{\partial B} = e^{-rT}\left[N[d_2] - \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{1}{2}d_2^2}\right] + (1-k)V_0 e^{\alpha T}\left[\frac{1}{B\sigma\sqrt{T}\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}\right]$. From (1b'), $d_1^2 = d_2^2 + 2d_2\sigma\sqrt{T} + \sigma^2 T$. After substituting this into the prior equation, using the definition of d_2^2 from (1b'), and after completing the squares in the exponents, I end up with $\frac{\partial D_0}{\partial B} = e^{-rT}\left[N[d_2] - \frac{1}{\sigma\sqrt{T}}n(d_2)\right] + (1-k)e^{-rT}\frac{1}{\sigma\sqrt{T}}n(d_2) = e^{-rT}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]$.

To prove the existence and uniqueness of a finite B_k^* when k>0 that satisfies $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 0$, I note that $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$ is everywhere continuous, $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) = 1$ for B=0 and that $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) \to 0$ as $B \to \infty$. Now I claim that $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} \to 0$ from below (i.e., the quantity is negative for B large). For this to be true, $\frac{k}{\sigma\sqrt{T}} > \frac{N[d_2]}{n(d_2)}$ for any k>0 as B gets large. Applying L'Hospital's rule to the RHS of the inequality shows that it goes to zero in the limit, confirming that $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) \to 0$ from below. Next, take the derivative of $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2)$ with respect to B, which results in $n(d_2)\frac{\partial d_2}{\partial B}\left[1 + d_2\frac{k}{\sigma\sqrt{T}}\right]$. The terms outside the bracket together are negative. The term inside the bracket is initially positive for B small and then eventually turns negative for some B sufficiently large. This implies that the slope of $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$ is initially negative as a function of B, but then turns positive for some unique B sufficiently large, and then stays positive thereafter. This is all that is needed for existence and uniqueness of B_k^*, since, for the above collection of facts to be true, it must be the case that there is a single crossing at zero in which there is one and only one B

for which $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 0$. Finally, given these facts, in the range of $B \in [0, B_k^*)$ it immediately follows that $\frac{\partial D_0}{\partial B} > 0$ and $\frac{\partial^2 D_0}{\partial B^2} < 0$ when k>0. QED

Proof of Proposition 2: From equation (2b) the FOC is: $\frac{\partial \Phi^V}{\partial B} = \rho\left[V_0 e^{(\mu+\alpha)T} n(h_1)\frac{\partial h_1}{\partial B} - N[h_2]\frac{\partial \chi_0}{\partial B} - \chi_0 n(h_2)\frac{\partial h_2}{\partial B}\right] = 0$. Recalling that $\chi_0 = B + (V_0 - D_0)e^{\psi T}$, it follows that $\frac{\partial \chi_0}{\partial B} = 1 - e^{(\psi - r)T}\left[N[d_2] - \frac{k}{\sigma\sqrt{T}} n(d_2)\right]$, $\psi \geq r$. Subbing this into the FOC and utilizing results from proposition 1, as well as well-known comparative static relations for call options with respect to B (the exercise price), the FOC simplifies to $\frac{\partial \Phi^V}{\partial B} = -\rho N[h_2]\left[1 - e^{(\psi - r)T}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]\right] = 0$. For ρ>0 the FOC comes down to equating the terms inside the brackets to zero. Existence and uniqueness follow based on the same logic spelled out in the proof to proposition 1. The lack of dependence on ρ is based on inspection of the FOC above. Finally, when $\psi < r$, inspection of $\frac{\partial \Phi^V}{\partial B}$ reveals that an internal optimum does not exist. Inspection further reveals that Φ^V is universally decreasing in B within the feasible range, implying B^*=0. QED

Proof of Corollary 2 to Proposition 2: $\psi = r - \frac{1}{T} ln\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]$ follows directly from the FOC derived for proposition 2. I will refer to this relation repeatedly to derive the comparative static results. The comparative static $\frac{\partial B^*}{\partial \psi} > 0$ follows from $\frac{\partial \psi}{\partial B} > 0$ in the above functional relation, since $N[d_2] - \frac{k}{\sigma\sqrt{T}} n(d_2)$>0 and the derivative of this quantity is negative, per the proof of proposition 1. I will use the fact that $\frac{\partial B^*}{\partial \psi} > 0$ and implicit differentiation to generate the other stated comparative static relations. In the case of $\frac{\partial B^*}{\partial k}$, inspection of the above relation reveals that $\frac{\partial \psi}{\partial k} > 0$, implying that $\frac{\partial B^*}{\partial k} < 0$. In the case of $\frac{\partial B^*}{\partial \alpha}$, $\frac{\partial \psi}{\partial \alpha}$ is seen to be negative, implying that $\frac{\partial B^*}{\partial \alpha} > 0$. For the cases of $\frac{\partial B^*}{\partial \sigma}$, $\frac{\partial B^*}{\partial T}$ and $\frac{\partial B^*}{\partial r}$, after quite a bit of tedious algebra I am unable to sign $\frac{\partial \psi}{\partial \sigma}$, $\frac{\partial \psi}{\partial T}$ and $\frac{\partial \psi}{\partial r}$. This in turn implies that I am unable to sign $\frac{\partial B^*}{\partial \sigma}$, $\frac{\partial B^*}{\partial T}$ and $\frac{\partial B^*}{\partial r}$. QED

Proof of Proposition 3: Given the Lagragian stated in (11), after examining necessary FOCs I find that, regardless of whether the fund leverage constraint is binding, $\mu_1 = \frac{\frac{-\partial \Phi^V}{\partial \rho}}{\frac{-\partial \Phi^V}{\partial \rho}\left(\frac{1}{T}\right)\left(\frac{1}{\mathcal{E}_T - \Phi^V}\right)} = T[\mathcal{E}_T - \Phi^V] > 0$. This implies that the fund target return constraint always binds. As a result, one finds $\bar{\rho} > 0$ as defined in equation (10), where positive $\bar{\rho}$ is necessary to satisfy GP participation. With μ_1, in the case of a binding fund leverage constraint, after some algebra I obtain $\mu_2 = \frac{1}{\frac{\partial D_0}{\partial B}}\left[\frac{\partial \mathcal{E}_T}{\partial B} - \left(\frac{\mathcal{E}_T - \Phi^V}{E_0 + \Phi^F}\right)\frac{\partial E_0}{\partial B}\right]$. The term inside the bracket must be verified to be positive for $\mu_2 > 0$. When $\mu_2 > 0$ it must be that $\frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}} > \frac{E_0 + \Phi^F}{\mathcal{E}_T - \Phi^V}$. Recalling equation (7), this inequality implies that $\frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}} > e^{-\lambda^* T}$. Also, GP participation vis-à-vis equation (10) implies that $\mathcal{E}_T - [E_0 + \Phi^F]e^{\lambda^* T} > 0$, which in turn implies that $e^{-\lambda^* T} > \frac{E_0 + \Phi^F}{\mathcal{E}_T}$. Altogether we have that $\frac{E_0 + \Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$

when $\mu_2 > 0$ and GP participation is satisfied as stated in the proposition. Furthermore, $\frac{E_0+\Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$ implies $\mu_2 > 0$. All terms are independent of ψ, as claimed. QED

<u>Proof of Proposition 4</u>: As a first step it is useful to write out the $\frac{\partial \Phi_{CU}^V}{\partial B}$, which can be expressed as $\frac{\partial \Phi_{CU}^V}{\partial B} = \xi[N(m_2) - N(h_2)]\left[1 - e^{(\psi-r)T}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]\right] - \rho N(m_2)$. Note that the bracketed term to the far left is always negative in the relevant range for *B*, as is the last term. When the larger bracketed term (that also contains the choke condition term) is positive, then $\frac{\partial \Phi_{CU}^V}{\partial B} < 0$. This is always true when $\psi \leq r$, and holds for all $B \geq 0$. It follows because $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 1$ at $B=0$, which then decreases in the feasible range $B \in [0, B_k^*]$. In this case $B_{CU}^* = 0$ as claimed. In the case of $\psi > r$ and $1 - e^{(\psi-r)T} \geq \frac{\rho N[m_2]}{\xi[N[m_2] - N[h_2]]}$ at $B=0$, inspection of $\frac{\partial \Phi_{CU}^V}{\partial B}$ above reveals that $\frac{\partial \Phi_{CU}^V}{\partial B} < 0$ for all $B \geq 0$, implying that $B_{CU}^* = 0$ in this case as well. Lastly, in the case of $\psi > r$ and $1 - e^{(\psi-r)T} < \frac{\rho N[m_2]}{\xi[N[m_2] - N[h_2]]}$ at $B=0$, inspection of $\frac{\partial \Phi_{CU}^V}{\partial B}$ above shows that the larger bracketed term is negative and that $\frac{\partial \Phi_{CU}^V}{\partial B} > 0$ at *B=0.* Now, because the RHS of the FOC expressed in equation (14b) is always negative, and because the LHS of (14b) is increasing in *B,* there will exist a $B_{CU}^* < B^*$ that satisfies the FOC written in (14b). QED

<u>Proof of Corollary to Proposition 4</u>: After differentiating Φ_{CU}^V with respect to ξ and doing some algebra, I obtain $\frac{\partial \Phi_{CU}^V}{\partial \xi} = \left[V_0 e^{(\mu+\alpha)T} N[h_1] - \chi_0 N[h_2]\right] - \left[V_0 e^{(\mu+\alpha)T} N[m_1] - \chi_0 N[mh_2]\right]$, which can be re-expressed as $\int_{\chi_0}^{\mathrm{X}_0} (V_T - \chi_0) f(V_T) dV_T > 0$ given that $\chi_0 < \mathrm{X}_0$. As for $\frac{\partial \lambda}{\partial \xi}$, differentiate lambda as defined in equation (8) with respect to ξ, noting that only Φ_{CU}^V depends on ξ. QED

<u>Proof of Proposition 5</u>: From the corollary to proposition 4, GP fees increase in the catch-up rate, while LP return decreases. Also, for parameter values considered herein, LP returns increase in fund leverage at and below the fund leverage constraint. Altogether this implies that there exists a minimum $\dot{\alpha} > 0$ at which $\xi = \rho$ with $\lambda = \lambda^*$ and $D_0 = \overline{D}_0$. Because LP return increases in *α*, any $\alpha \leq \dot{\alpha}$ implies that the catch-up fee provision cannot be implemented with $\xi > \rho$ while simultaneously satisfying the target return and fund leverage constraints. This establishes the <u>lower range of *α*'s</u>.

Next, I will identify an $\ddot{\alpha} > \dot{\alpha}$ such that $\xi = 1.0$ with $\lambda = \lambda^*$ and $D_0 = \overline{D}_0$. This establishes an upper bound for the <u>middle range of *α*'s</u>. I now claim that for $\dot{\alpha} < \alpha < \ddot{\alpha}$, the target return and fund leverage constraints bind with ξ chosen to satisfy equation (7a). This claim requires that $\mu_1, \mu_2 > 0$ for there to be a constrained optimum. After examining FOCs that follow from equation (12), I find $\mu_1 = T[\mathcal{E}_T - \Phi_{CU}^V] > 0$ and $\mu_2 = \frac{1}{\frac{\partial D_0}{\partial B}}\left[\frac{\partial \mathcal{E}_T}{\partial B} - \left(\frac{\mathcal{E}_T - \Phi_{CU}^V}{E_0 + \Phi^F}\right)\frac{\partial E_0}{\partial B}\right]$. These are precisely the same conditions that are required to hold in the constrained baseline contract problem characterized in proposition 3, where the only difference is that I am optimizing the expanded contract with respect to the catch-up rate, ξ, instead of the baseline contract with respect to the carry share percentage, ρ. μ_1 is seen to be always positive, while the bracketed term of μ_2 must be verified as positive given the parameter set in question.

Lastly, $\alpha \geq \ddot{\alpha}$ defines the <u>higher range of *α*'s</u>. In this case I claim that the target return and full catch rate constraints bind. Here D_0 is positive but less than $\overline{D}_0$, with D_0 determined by (7a). For this claim to hold, I must verify that $\mu_1,\mu_3>0$. Solving the constrained optimization problem in (12) generates that $\mu_1 = \frac{\frac{-\partial \Phi_{CU}^V}{\partial B}}{\frac{\partial \lambda^N}{\partial B}}$ and $\mu_3 = \frac{\partial \Phi_{CU}^V}{\partial \xi}\left[1 - \frac{\mu_1}{T[\mathcal{E}_T - \Phi_{CU}^V]}\right]$. These KKT conditions are verified to hold for empirically supported parameter ranges applied herein. QED

Appendix B

Alternative Model of Private Equity Fundraising

With the alternative fundraising model, I fix $E_0 = \bar{E}_0$ and ask how large V_0 should be given that debt finances all fund acquisitions in excess of $\bar{E}_0$. Consequently, the PE fundraising game now has two independent stages, with equity fundraising coming in the first stage and optimal fund size determination based on debt financing in the second stage.

This approach to fundraising complicates debt valuation, because debt is now self-referencing. That is, $\breve{D}_0 = e^{-rT} BN[\breve{d}_2] + (1-k)\breve{V}_0 e^{\alpha T} N[-\breve{d}_1]$ as before, with $\breve{d}_1 = \frac{ln\left[\breve{V}_0/_B\right] + \left((r+\alpha)+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, $\breve{d}_2 = \breve{d}_1 - \sigma\sqrt{T}$. But now $\breve{V}_0 = \bar{E}_0 + D_0$ and $\breve{D}_0 = D_0$ are imposed as constraints. I note that $\breve{D}_0$ is well-behaved (continuous and increasing) as it depends on B for $B \in [0, B_k^*)$.

Taking the total derivative of $\breve{D}_0$ with respect to B, I obtain $\frac{d\breve{D}_0}{dB} = \frac{\partial \breve{D}_0}{\partial B} + \frac{\partial \breve{D}_0}{\partial \breve{V}_0}\frac{\partial \breve{V}_0}{\partial B}$. Since $\breve{V}_0 = \bar{E}_0 + D_0$ and $\breve{D}_0 = D_0$, $\frac{d\breve{D}_0}{dB} = \frac{\partial \breve{V}_0}{\partial B}$ and the total derivative can be rewritten as $\frac{d\breve{D}_0}{dB} = \frac{\partial D_0}{\partial B}\left[1 + \frac{\partial \breve{D}_0}{\partial \breve{V}_0}\right]$. Since $\breve{V}_0 = \bar{E}_0 + D_0$, it is clear that $\frac{\partial \breve{D}_0}{\partial V_0} > 0$ in the relevant range for B. In particular, $\frac{\partial \breve{D}_0}{\partial V_0} = (1-k)N[-\breve{d}_1] + n(\breve{d}_1)\frac{k}{\sigma\sqrt{T}} > 0$. Thus, not only is $\frac{d\breve{D}_0}{dB}$ positive, but $\frac{d\breve{D}_0}{dB} > \frac{\partial D_0}{\partial B}$ for $B \in [0, B_k^*)$.

With this result I am now in a position to consider the GP's problem of optimizing fund size as it depends on B. As before, the problem is stated as: $\underset{B}{Max}\ \Phi^V(B;\psi,\rho) = \rho\left[\breve{V}_0 e^{(\mu+\alpha)T} N[\breve{h}_1] - \chi_0 N[\breve{h}_2]\right]$, $\breve{h}_1 = \frac{ln\left[\breve{V}_0/_{\chi_0}\right] + \left((\mu+\alpha)+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, $\breve{h}_2 = \breve{h}_1 - \sigma\sqrt{T}$. Further, as above, $\breve{V}_0$ is a function of B, with the previously imposed constraints applying. Evaluating incentive compatibility results in the following relation: $\frac{1}{\frac{d\breve{D}_0}{dB}} = \frac{e^{(\mu+\alpha)T} N[\breve{h}_1]}{N[\breve{h}_2]} > e^{(\mu+\alpha)T} > e^{\psi T}$ when $\mu + \alpha > \psi$. A unique interior solution therefore exists when $r<\mu+\alpha$, which is always the case as long as $\mu>r$ and $\alpha\geq 0$. Since $\frac{d\breve{D}_0}{dB} > \frac{\partial D_0}{\partial B}$ for $B \in [0, B_k^*)$, the marginal cost of debt in this case (LHS of above relation) is less than the marginal cost of debt with the baseline fundraising model. This implies that the optimal B is larger is this fundraising model whenever $\mu + \alpha > \psi$. And since $\frac{d\breve{D}_0}{dB} > \frac{\partial D_0}{\partial B}$, the optimal $\breve{D}_0$ will also be larger than D_0. Lastly, note that $\breve{D}_0 = 0$ when $D_0 = 0$, implying that the marginal cost of debt increases without bound as D_0 approaches zero. This in turn implies that the optimal $\breve{D}_0$ is finite and therefore fund size is finite.